Polymorphisms of small digraphs, and the complexity of CSP

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Outline

The aim: digraphs with interesting polymorphisms

Results:

- the list of interesting polymorphisms of all digraphs up to 5 vertices (by means of massive computation)
- the smallest digraph with currently unknown complexity of the retraction problem

Outline of the talk:

1. Motivation = introduction to the complexity of CSP
2. Results of the computation

CSP: a computer science description

INPUT: variables, domain, constraints
OUTPUT: assign domain elements to the variables so that the constraints are satisfied

Examples:

- SAT: assign 0,1 to variables so that given clauses are satisfied
- k-coloring: assign colors to vertices so that no adjacent ones have the same color
- solving equations over finite fields: assign elements to variables so that given equations are satisfied
- industry: scheduling, etc.

Complexity:

- NP-complete
- if constraints are restricted, may fall into P

CSP($\mathbb{B}$): a combinatorial description

Fix a finite relational structure $\mathbb{B}$.

INPUT: relational structure $\mathbb{A}$
OUTPUT: find a homomorphism $\mathbb{A} \to \mathbb{B}$

Examples:

- 3-SAT: $\mathbb{B} = (\{0,1\}, \rho_{000}, \ldots, \rho_{111})$ where $\rho_{ijk} = \{0,1\}^3 \setminus \{(i,j,k)\}$
- k-coloring: $\mathbb{B} = \mathbb{K}_k$ (complete graph)
- linear equations over $\mathbb{Z}_p$: $\mathbb{B} = (\{0, \ldots, p-1\}, \rho, \{1\})$ where $(a, b, c) \in \rho$ iff $a + b = c$

Complexity:
Dichotomy Conjecture (Feder, Vardi, 1999):
For every $\mathbb{B}$, CSP($\mathbb{B}$) is in P, or is NP-complete

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A technical remark: CSP($\mathbb{B}$)=CSP(core of $\mathbb{B}$).
From now on, all structures are assumed to be *cores*.

The algebraic approach to complexity of CSP

The complexity of CSP($\mathbb{B}$) depends on *idempotent polymorphisms* of $\mathbb{B}$.

- f is a *polymorphism* of $\mathbb{B} = (B, R)$, if f preserves every $\rho \in R$. Equivalently, f is a homomorphism $\mathbb{B}^n \to \mathbb{B}$.
- f is *idempotent*, if $f(x, \dots, x) = x$.

Complexity:

Algebraic Dichotomy Conjecture (Bulatov, Jeavons, Krokhin, 2005):
$\mathbb{B}$ has *nice* idempotent polymorphisms $\Rightarrow$ CSP($\mathbb{B}$) is in P
$\mathbb{B}$ has *no nice* idempotent polymorphisms $\Rightarrow$ CSP($\mathbb{B}$) is NP-complete

- the former has been confirmed for many special cases (e.g., smooth digraphs, or 3-element structures)
- the latter is a theorem

Fact: The nicer polymorphisms $\mathbb{B}$ has, the easier CSP($\mathbb{B}$) is.

Polymorphism conditions

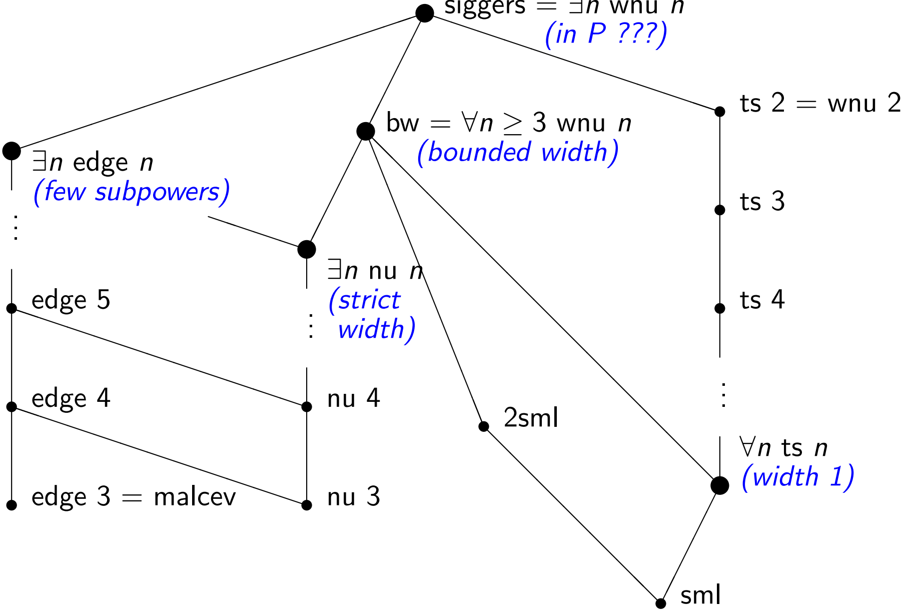
Some nice polymorphisms:

- weak near-unanimity:
  $f(yxx\dots x) = f(xyx\dots x) = \dots = f(xxx\dots xy)$
- near-unanimity polymorphism:
  $f(yxx\dots x) = f(xyx\dots x) = \dots = f(xxx\dots xy) = x$
- edge: $f(yyxx\dots x) = f(yxyx\dots x) = x$ and
  $f(xxyx\dots x) = f(xxxyx\dots x) = \dots = f(xxx\dots xy) = x$
- totally symmetric: $f(x_1, x_2, \dots, x_n) = f(y_1, y_2, \dots, y_n)$ whenever $\{x_1, \dots, x_n\} = \{y_1, \dots, y_n\}$
- semilattice: $f(x, y) = f(y, x)$ and $f(f(x, y), z) = f(x, f(y, z))$
- 2-semilattice $f(x, y) = f(y, x)$ and $f(f(x, y), x) = f(x, y)$

The weakest polymorphism conditions:

- (Maróti, McKenzie) n-ary WNU for some n
- (Barto, Kozik) n-ary cyclic polymorphism for some n
- (Siggers) a polymorphism satisfying $f(x, y, y, z) = f(y, x, z, x)$

Poset of polymorphism conditions



Our setting

Reductions

- (Feder, Vardi) For every $\mathbb{B}$, there is a digraph $\mathbb{G}$ such that CSP($\mathbb{B}$) is poly-equivalent to CSP($\mathbb{G}$).
- we can add constants, i.e., consider $\bar{\mathbb{G}} = (V, E, \{v\} : v \in V)$
  - it has the same clone of polymorphisms
  - it is a core
  - CSP($\bar{\mathbb{G}}$) = *retraction problem* for $\mathbb{G}$,
for cores it is poly-equivalent to CSP($\mathbb{G}$)

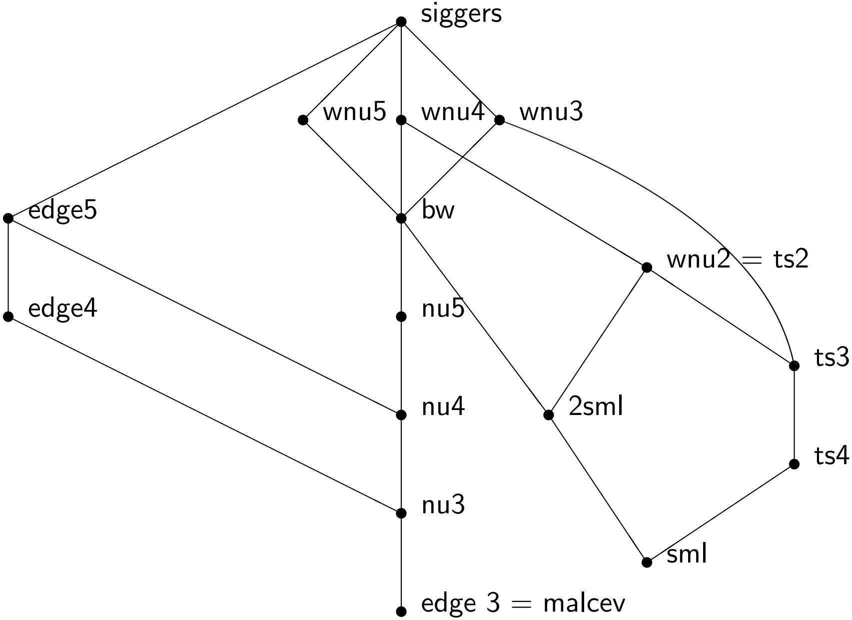
Our job:

to determine interesting polymorphism conditions for

- all digraphs with 2 to 5 vertices,
- many random digraphs on up to 8 vertices

Note: it is sufficient to determine *minimal* conditions.

Interesting polymorphisms



2-element digraphs

#	table	minimal conditions
01	00 00	malcev sml
02	01 00	malcev sml
03	01 10	malcev
04	00 01	malcev sml
05	01 01	malcev sml
06	00 11	malcev sml
07	01 11	nu3 sml
08	10 01	malcev sml
09	11 01	nu3 sml
10	11 11	malcev sml

(Interested in tables for larger digraphs? See our website!)

The number of digraphs with given minimal condition

condition	size 2	size 3	size 4	size 5
NONE	0	7	765	155151
malcev	8	29	118	471
nu3	2	63	1572	60056
nu4	0	0	46	8916
nu5	0	0	1	1388
edge4	0	0	0	0
edge5	0	0	0	0
bw	0	0	29	3475
sml	9	90	2178	130870
2sml	0	1	12	1639
ts4	0	0	6	1559
ts3	0	0	0	0
wnu2,3,4,5	0	0	0	0
siggers	0	0	0	0
TOTAL	10	104	3044	291968

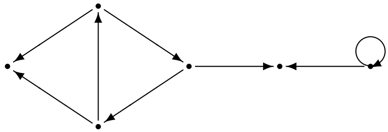
The number of digraphs with given minimal conditions

conditions	size 2	size 3	size 4	size 5
malcev sml	7	24	92	358
nu3 sml	2	61	1532	59281
malcev	1	4	22	89
sml	0	5	507	60931
nu3	0	2	38	745
malcev 2sml	0	1	4	23
nu4 sml	0	0	46	8914
bw	0	0	29	3475
2sml ts4	0	0	6	1556
nu5 sml	0	0	1	1386
nu3 2sml	0	0	2	30
2sml	0	0	0	25
nu5 2sml ts4	0	0	0	2
nu4 2sml	0	0	0	2
malcev 2sml ts4	0	0	0	1
NONE	0	7	765	155151

Conclusions

On ≤ 5 vertices, each digraph is either NP-complete, or has bounded width (and thus is in P).

On 6 vertices: consider the *sea devil* graph:



It has 3-ary and 5-ary WNU, but no 4-ary WNU.
Hence, conjecturally is in P, but fails bounded width.

Open problem:

- is *sea devil* in P ?
  (the smallest digraph with currently unknown complexity of CSP)
- particularly, is *sea devil* solvable by few subpowers?
  (we computed: no edge polymorphism up to arity 7)