Identities with squares of binomial coefficients

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Abstract
This paper introduces a method for finding closed forms for certain sums involving squares of binomial coefficients. We use this method to present an alternative approach to a problem of evaluating a different type of sums containing squares of the numbers from Catalan’s triangle.

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1 Introduction
In the first part of our paper, we present a method for finding closed forms for the sums

\[ S_k(n) := \sum_{l=0}^{n-1} k^l \binom{2n}{l}^2, \quad n \geq 1, \]

\[ T_k(n) := \sum_{l=0}^{n-2} k^l \binom{2n-1}{l}^2, \quad n \geq 2 \]

(we use the convention \(0^0 = 1\), where \(k \geq 0\) is a fixed integer. These sums are somewhat tricky in the sense that the standard techniques for hypergeometric summation (see [5]) are not applicable. Indeed, the summation does not run over all possible values of \(l\), which means that methods like Zeilberger’s algorithm or Sister Celine’s method cannot be used; Gosper’s algorithm for indefinite summation fails, too.

In the second part of the paper, we apply our results to evaluate certain sums involving squares of the numbers from Catalan’s triangle.

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2 Calculating $S_k(n)$ and $T_k(n)$

The formulas for $S_k(n)$ and $T_k(n)$ in the cases $k = 0$ and $k = 1$ are well-known:

$$S_0(n) = \sum_{l=0}^{n-1} \binom{2n}{l}^2 = \frac{1}{2} \left( \frac{4n}{2n} - \binom{2n}{n} \right), \quad n \geq 1,$$

$$T_0(n) = \sum_{l=0}^{n-2} \binom{2n-1}{l}^2 = \frac{1}{2} \left( \frac{4n-2}{2n-1} - \binom{2n-1}{n-1} \right), \quad n \geq 2,$$

$$S_1(n) = \sum_{l=0}^{n-1} l \binom{2n}{l}^2 = n \left( \frac{4n-1}{2n-1} - 3 \binom{2n-1}{n-1} \right), \quad n \geq 1,$$

$$T_1(n) = \sum_{l=0}^{n-2} l \binom{2n-1}{l}^2 = \frac{2n-1}{2} \left( \frac{4n-3}{2n-2} - 2 \binom{2n-1}{n-1} \binom{2n-2}{n-2} - \binom{2n-2}{n-1} \right), \quad n \geq 2.$$  

These identities follow easily (see e.g. [3]) from the standard formula

$$\sum_{i=0}^{k} \binom{k}{i}^2 = \binom{2k}{k},$$

and the “absorption” identity

$$\binom{k}{i} = \frac{k}{i} \binom{k-1}{i-1}, \quad i \geq 1.$$

The following theorem shows that for $k \geq 2$, there is a formula for $S_k$ which depends on $T_0, \ldots, T_{k-2}$:

**Theorem 1.** If $k \geq 2$, then

$$S_k(n) = 4n^2 \sum_{i=0}^{k-2} \binom{k-2}{i} T_i(n), \quad n \geq 1.$$  

**Proof.** We use the absorption identity and binomial theorem:

$$S_k(n) = \sum_{l=1}^{n-1} l^k \binom{2n}{l}^2 = 4n^2 \sum_{l=1}^{n-1} l^{k-2} \binom{2n-1}{l-1}^2 = \ldots$$
\[
4n^2 \sum_{l=0}^{n-2} (l+1)^{k-2} \binom{2n-1}{l}^2 = \\
4n^2 \sum_{l=0}^{n-2} \sum_{i=0}^{k-2} \binom{k-2}{i} l^i \binom{2n-1}{l}^2 = 4n^2 \sum_{i=0}^{k-2} \binom{k-2}{i} T_i(n).
\]

There is a similar theorem which gives a formula for \( T_k, k \geq 2, \) in terms of \( S_0, \ldots, S_{k-2} \):

**Theorem 2.** If \( k \geq 2 \), then

\[
T_k(n) = (2n-1)^2 \sum_{i=0}^{k-2} \binom{k-2}{i} \left( S_i(n-1) - (n-2)^i \binom{2n-2}{n-2} \right), \quad n \geq 2.
\]

**Proof.**

\[
T_k(n) = \sum_{l=1}^{n-2} l^k \binom{2n-1}{l}^2 = (2n-1)^2 \sum_{l=1}^{n-2} \binom{k-2}{l-1} \binom{2n-2}{l-1}^2 = \\
(2n-1)^2 \sum_{l=0}^{n-3} (l+1)^{k-2} \binom{2n-2}{l}^2 = \\
(2n-1)^2 \sum_{l=0}^{n-3} \sum_{i=0}^{k-2} \binom{k-2}{i} l^i \binom{2n-2}{l}^2 = \\
(2n-1)^2 \sum_{i=0}^{k-2} \binom{k-2}{i} \left( \sum_{l=0}^{n-2} l^i \binom{2n-2}{l}^2 - (n-2)^i \binom{2n-2}{n-2} \right) = \\
(2n-1)^2 \sum_{i=0}^{k-2} \binom{k-2}{i} \left( S_i(n-1) - (n-2)^i \binom{2n-2}{n-2} \right).
\]

Since the formulas for \( S_0, S_1, T_0 \) and \( T_1 \) are known, the previous two theorems can be used to calculate \( S_k \) and \( T_k \) successively for every \( k \). For example, we obtain

\[
S_2(n) = 4n^2 \left( \frac{1}{2} \binom{4n-2}{2n-1} - \binom{2n-1}{n-1} \right)
\]
(this agrees with the formula given in [3]),
\[
T_2(n) = (2n - 1)^2 \left( \frac{1}{2} \left( \left( \frac{4n - 4}{2n - 2} \right) - \left( \frac{2n - 2}{n - 1} \right)^2 \right) - \left( \frac{2n - 2}{n - 2} \right)^2 \right).
\]

The complexity of the formulas for \( S_k \) and \( T_k \) grows with increasing \( k \), but the calculations (including simplification) can be performed using a computer.

**Remark 3.** A similar approach may be used to evaluate more general sums of the form

\[
T^m_k(n) := \sum_{l=0}^{n-2} l^k \left( \frac{2n - m}{l} \right)^2, \quad n \geq m/2,
\]

where \( m \) is an arbitrary fixed positive or negative integer and \( k \geq 0 \) as before; note that \( T_k = T^1_k \). If \( m \) is even, then

\[
T^m_k(n) = \sum_{l=0}^{n-m/2-1} l^k \left( \frac{2(n - m/2)}{l} \right)^2 = \sum_{l=n-m/2}^{n-2} l^k \left( \frac{2(n - m/2)}{l} \right)^2 = S_k(n - m/2) + \sum_{p=0}^{m/2-2} (p + n - m/2)^k \left( \frac{2(n - m/2)}{p + n - m/2} \right)^2
\]

(we use the convention \( \sum_{i=a}^{b} f(i) = -\sum_{i=a}^{b} f(i) \) if \( a > b \)). Otherwise, if \( m \) is odd, then

\[
T^m_k(n) = (2n - m)^2 \sum_{l=1}^{n-2} l^{k-2} \left( \frac{2n - m - 1}{l - 1} \right)^2 = (2n - m)^2 \sum_{l=0}^{n-3} (l + 1)^{k-2} \left( \frac{2n - m - 1}{l} \right)^2 = (2n - m)^2 \sum_{l=0}^{n-3} k^{k-2} \sum_{p=0}^{2(n - m - 1)} \left( \frac{k - 2}{p} \right)^{p} \left( \frac{2n - m - 1}{l} \right)^2 = (2n - m)^2 \sum_{p=0}^{k-2} \frac{k - 2}{p} \left( \sum_{l=0}^{n-2} l^{p} \left( \frac{2n - m - 1}{l} \right)^2 \right) - (n - 2)^p \left( \frac{2n - m - 1}{n - 2} \right)^2
\]
\[(2n - m)^2 \sum_{p=0}^{k-2} \binom{k-2}{p} \left( T_{m}^{m+1}(n) - (n-2)^p \binom{2n-m-1}{n-2} \right)^2 ,\]

i.e. \( T_k^n \) can be expressed in terms of \( T_0^{m+1}, \ldots, T_{k-2}^{m+1} \), whose upper indices are even.

### 3 A sum related to Catalan’s triangle

Catalan’s triangle was introduced in [6] by L. W. Shapiro. Its entries are the numbers

\[ B_{n,l} := \frac{l}{n} \left( \binom{2n}{n-l} \right), \quad n, l \in \mathbb{N}, l \leq n. \]

The name of the triangle stems from the fact that

\[ B_{n,1} = \frac{1}{n} \left( \binom{2n}{n-1} \right) = \frac{1}{n+1} \left( \binom{2n}{n} \right) = C_n, \]

i.e. the Catalan numbers appear in the first column of the triangle.

The numbers \( B_{n,l} \) appear in several combinatorial problems and identities (see e.g. [6], [3], [2]). The authors of the paper [3] have pointed out a relation of these numbers to a problem of the dynamical behavior of a family of iterative methods applied to quadratic polynomials. They also mentioned the problem of evaluating the sums

\[ A_k(n) := \sum_{l=1}^{n} l^k B_{n,l}^2, \quad n \in \mathbb{N}, \]

where \( k \geq 0 \) is a fixed integer. They have obtained closed forms for \( A_0, A_1 \) and \( A_2 \); we need the first two formulas, which are true for \( n \geq 1 \):

\[ A_0(n) = \sum_{l=1}^{n} B_{n,l}^2 = C_{2n-1} = \frac{1}{2(4n-1)} \left( \frac{4n}{2n} \right), \]

\[ A_1(n) = \sum_{l=1}^{n} l B_{n,l}^2 = \frac{n(n+1)}{2} C_{n-1} C_n = \frac{n}{4(2n-1)} \left( \frac{2n}{n} \right)^2. \]

Additional closed forms for \( A_3, \ldots, A_7 \) were found and proved using the Wilf-Zeilberger method by the same authors in [4]. Finally, general formulas for an arbitrary \( A_n \) are given in [1]. We now show a different method of deriving the corresponding formulas.

The following theorem shows that for arbitrary \( k \), the sum \( A_k \) can be expressed in terms of \( S_0, \ldots, S_{k+2} \):
Theorem 4. If \( k \geq 0 \) and \( n \geq 1 \) are arbitrary integers, then
\[
A_k(n) = \sum_{j=0}^{k+2} \binom{k+2}{j} n^{k-j} (-1)^j S_j(n).
\]

Proof.
\[
A_k(n) = \sum_{m=1}^{n} m^k B_{n,m}^2 = \frac{1}{n^2} \sum_{m=1}^{n} m^{k+2} \left( \frac{2n}{n-m} \right)^2 = \frac{1}{n^2} \sum_{l=0}^{n-1} (n-l)^{k+2} \binom{2n}{l}^2 =
\]
\[
\frac{1}{n^2} \sum_{l=0}^{n-1} \sum_{j=0}^{k+2} \binom{k+2}{j} n^{k+2-j} (-1)^j \binom{2n}{l}^2 =
\]
\[
\sum_{j=0}^{k+2} \binom{k+2}{j} n^{k-j} (-1)^j \sum_{l=0}^{n-1} \binom{2n}{l}^2 = \sum_{j=0}^{k+2} \binom{k+2}{j} n^{k-j} (-1)^j S_j(n).
\]

This means that given a fixed \( k \geq 0 \), we have a method for obtaining a closed form for \( A_k \). To avoid laborious calculations, it’s again better to use a computer. All the following results were obtained using Mathematica. We start with the first few identities for \( A_k \), where \( k \) is even:
\[
A_2(n) = \frac{n(3n-2)}{2(4n-3)(4n-1)} \binom{4n}{2n},
\]
\[
A_4(n) = \frac{n (15n^3 - 30n^2 + 16n - 2)}{2(4n-5)(4n-3)(4n-1)} \binom{4n}{2n},
\]
\[
A_6(n) = \frac{n(105n^5 - 420n^4 + 588n^3 - 356n^2 + 96n - 10)}{2(4n-7)(4n-5)(4n-3)(4n-1)} \binom{4n}{2n},
\]
And now a few formulas for \( A_k \), where \( k \) is odd:
\[
A_3(n) = \frac{n^2}{4(2n-1)} \binom{2n}{n}^2,
\]
\[
A_5(n) = \frac{n^2 (3n^2 - 5n + 1)}{4(2n-3)(2n-1)} \binom{2n}{n}^2,
\]
\[
A_7(n) = \frac{n^2 (6n^3 - 12n^2 + 6n - 1)}{4(2n-3)(2n-1)} \binom{2n}{n}^2.
\]
Note that these formulas are written in a slightly different form than in [1] and [4]; our form has the advantage that the right-hand sides are defined for
every $n \geq 1$. It should be noted that with increasing $k$, the time needed to perform the calculation according to Theorem 4 grows rather quickly. The Mathematica code which was used to perform all the calculations presented in this paper is available from

www.karlin.mff.cuni.cz/~slavik/mathematica/identities.nb

(including more formulas which have been omitted because of their complexity).

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References


