Prague Strategies

Libor Barto

joint work with Marcin Kozik

Department of Algebra
Faculty of Mathematics and Physics
Charles University in Prague
Czech Republic

NSAC 2009
The Prague Theorem

<table>
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<th>Theorem (Barto, Kozik 2009)</th>
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The Prague Theorem

Theorem (Barto, Kozik 2009)

Let $A$ be an idempotent algebra. TFAE

- $A$ is an $SD(\land)$ algebra
  ($= \text{lies in a variety omitting 1 and 2}$)
- Every Prague strategy over $A$ has a solution

Plan:

- $k$-intersection property
- $SD(\land)$
- $CSP(A)$
- $(k, l)$-minimal instance
- Prague strategy
Warning

All algebras are finite and idempotent
$k$-intersection property

**Definition (k-equal relations)**

$R_1, R_2 \subseteq A^n$ are \textit{k-equal}, if $\forall J \subseteq [n], |J| \leq k$, the projections of $R_1$ and $R_2$ to $J$ are equal.
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**Definition (k-intersection property, Valeriote)**

A finite algebra \( A \) satisfies the \textit{k-intersection property}, if \( \forall n \) every collection of pairwise k-equal non-empty subuniverses \( R_1, \ldots, R_m \leq A^n \) has nonempty intersection.
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A finite algebra $A$ satisfies the **k-intersection property**, if $\forall n$ every collection of pairwise k-equal non-empty subuniverses $R_1, \ldots, R_m \subseteq A^n$ has nonempty intersection.

**Observation**

$B \in \text{HSP}(A)$. Then $A$ has the k-intersection property $\Rightarrow B$ has the k-intersection prop.
Observation

If $A$ is a reduct of a module and $|A| > 1$, then $A$ fails the $k$-intersection property for every $k$. 

Proof.

For $a \in A$, let $R_a = \{ (a_1, \ldots, a_{k+1}) : a_1 + a_2 + \cdots + a_{k+1} = a \}$. Clearly $R_a$ is a subuniverse of $A_{k+1}$, and any projection to less than $k+1$ coordinates is full. If $a \neq b$ then $R_a \cap R_b = \emptyset$. 

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\[ A \text{ has the } k\text{-intersection property} \Rightarrow B \text{ has the } k\text{-intersection prop.} \]

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### Corollary

If \( A \) has the \( k\)-intersection property for some \( k \), then \( HSP(A) \) doesn’t contain a reduct of a module (with more than one element).
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If \( A \) has the \( k \)-intersection property for some \( k \), then \( \text{HSP}(A) \) doesn’t contain a reduct of a module (with more than one element).

Conjecture (Valeriote)

The other implication is also true.
Theorem (Hobby, Maróti, McKenzie, Valeriote, Willard)

Let $A$ be an algebra. TFAE

- $\text{HSP}(A)$ doesn’t contain a reduct of a module ($>1$ element)
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$A$ is $\text{SD}(\land)$, if it satisfies the equivalent conditions above
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  if $B \in \text{HSP}(A)$, $\alpha, \beta_1, \beta_2 \in \text{Con}(B)$

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- If $A$ has a Jónsson chain of terms, then $A$ is $\text{SD}(\wedge)$

Facts about intersection properties

- If $A$ has a semilattice term, then $A$ has the 1-intersection property
- If $A$ has a $k$-ary near-unanimity term, then $A$ has the $(k-1)$-intersection property Baker, Pixley
- If $A$ has a short (3-terms) chain of Jónsson terms, then $A$ has the 2-intersection property Kiss, Valeriote and 2 is the optimal number
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Let \( A \) be an algebra. An instance of CSP(\( A \)) is a pair \( (V, C) \), where

- \( V \) is a finite set (elements are called variables)
- \( C \) is a finite set of constraints

Constraint is a subuniverse \( C \) of \( A^D \), where \( D \subseteq V \) (called the scope of \( C \))
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A solution to \((V, C)\) is a mapping \(f : V \rightarrow A\) which satisfies all the constraints \(C \leq A^D\) in \(C\), i.e. \(f|D \in C\).
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Definition

A solution to \((V, C)\) is a mapping \(f : V \rightarrow A\) which satisfies all the constraints \(C \leq A^D\) in \(C\), i.e. \(f|D \in C\).

The aim is to find a solution fast (in poly-time).
Definition (± Bulatov, Jeavons)

Let $k \leq l$ be natural numbers. An instance $(V, C)$ of CSP($A$) is called $(k, l)$-minimal if

- Every $l$-element subset of $V$ is a subset of the scope of some constraint in $C$.
- For every $J \subseteq V$, $|J| \leq k$ and every pair $C_1, C_2 \in S$ whose scopes contain $J$, the projections of $C_1$ and $C_2$ onto $J$ are the same.

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(\(k, l\))-minimal instance

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An instance \((V, C)\) is called \(k\)-minimal, if it is \((k, k)\)-minimal.

**Observation**

If \(k' \leq k\) and \(l' \leq l\) then \((k, l)\)-minimal instance is \((k', l')\)-minimal.
Bounded relational width

Observation

Every instance of $\text{CSP}(A)$ can be converted into an equivalent $(k, l)$-minimal instance in poly-time.

(Two instances are equivalent if they have the same set of solutions.)
Bounded relational width

Observation
Every instance of CSP(\(A\)) can be converted into an equivalent \((k, l)\)-minimal instance in poly-time.
(Two instances are equivalent if they have the same set of solutions.)

Definition
\(A\) has relational width \((k, l)\) if every \((k, l)\)-minimal instance, whose constraints are non-empty, has a solution.
\(A\) has bounded relational width if it has relational width \((k, l)\) for some \(k, l\).
The bounded relational width conjecture

Theorem (Larose, Zádori, Bulatov)

If $A$ has bounded relational width, then $A$ is an $\text{SD}(\wedge)$ algebra.

Conjecture (Larose, Zádori, Bulatov)

The other implication is also true.

▶ If $A$ has a semilattice term, then $A$ has rel. width 1 Feder, Vardi, Dalmau, Pearson

▶ If $A$ has a 2-semilattice term, then $A$ has rel. width 3 Bulatov

▶ If $A$ has a $k$-ary near-unanimity term, then $A$ has rel. width $k-1$ Feder, Vardi

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▶ If $A$ has a short chain of Jónsson terms (4 terms), then $A$ has "bounded width" Carvalho, Dalmau, Marković, Maróti
**The bounded relational width conjecture**

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- If $A$ has a semilattice term, then $A$ has rel. width 1 [Feder, Vardi, Dalmau, Pearson]
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Corollaries of the Prague Theorem

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Corollary

If $A$ is an $SD(\wedge)$ algebra, then $A$ has relational width $(2, 3)$. (The parameters $(2, 3)$ are optimal.)

Corollary

If $A$ is an $SD(\wedge)$ algebra, then $A$ satisfies the 2-intersection property. (Recall that 2 is optimal.)
Corollaries of the Prague Theorem

**Corollary**

If $A$ is an $SD(\wedge)$ algebra, then $A$ has relational width $(2, 3)$. (The parameters $(2, 3)$ are optimal.)

**Corollary**

If $A$ is an $SD(\wedge)$ algebra, then $A$ satisfies the 2-intersection property. (Recall that 2 is optimal.)

**Proof.**

- Let $R_1, \ldots, R_m \leq A^n$ be nonempty and 2-equal
- Let $V = [n]$, $C = \{R_1, \ldots, R_m\}$
- $(V, C)$ is a $(2, n)$-minimal instance of $\text{CSP}(A)$
I am finally going to introduce Prague strategies.

Comparison with known notions:

\[(2, 3)\text{-minimal instance of } \text{CSP}(A)\]
\[\Downarrow\]
Prague strategy over \(A\)
\[\Downarrow\]
1-minimal instance of \(\text{CSP}(A)\)
Let \((V, \mathcal{C})\) be an instance of \(\text{CSP}(A)\)
For $x, y \in V$ and $C \in \mathcal{C}$ and $a, b \in A$ we write $a \xrightarrow{x,y,C} b$, if

- $x, y$ are in the scope of $C$
- The mapping $x \rightarrow a, y \rightarrow b$ is in the projection of $C$ to $\{x, y\}$
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**Definition**

A *pattern* $w$ is a tuple $(x_1, C_1, \ldots)$:

$$x_1 \xrightarrow{C_1} x_2 \xrightarrow{C_2} \ldots \xrightarrow{C_i} x_{i+1},$$

where $x_j \in V$ and $C_j \in \mathcal{C}$.

We write $a \xrightarrow{w} b$, if there exist $a = a_1, a_2, \ldots, a_{i+1} = b$ such that

$$a = a_1 \xrightarrow{x_1,x_2,C_1} a_2 \xrightarrow{x_2,x_3,C_2} a_3 \rightarrow \cdots \rightarrow a_i \xrightarrow{x_i,x_{i+1},C_i} a_{i+1} = b$$

The **scope** of $w$ is $[[w]] = \{x_1, \ldots, x_{i+1}\}$
If patterns $w_1, w_2$ start and end with the same variable $x$, we can form their concatenation $w_1 \circ w_2$.

$$w^K = w \circ w \circ \cdots \circ w \ (K\text{-times})$$
A Prague strategy over $\mathbf{A}$ is an instance $(V, C)$ of $\text{CSP}(\mathbf{A})$ such that

1. $(V, C)$ is 1-minimal
2. For every $x \in V$,
   - every pattern $v$ starting and ending with $x$,
   - every $a, b \in A$ such that $a \xrightarrow{v} b$ and every pattern $w$ starting and ending with $x$ s.t. $[[v]] \subseteq [[w]]$, there exists a natural number $K$ such that $a \xrightarrow{w^K} b$
Definition (!!!!!!!!)

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   - every pattern $w$ starting and ending with $x$ s.t. $[[v]] \subseteq [[w]]$,
   
   there exists a natural number $K$ such that $a \xrightarrow{w^K} b$

Observation

Every $(2, 3)$-min. instance of $\text{CSP}(\mathbf{A})$ is a Prague strategy over $\mathbf{A}$. 
The Prague Theorem

Theorem (BK)

Let $A$ be an algebra. TFAE

- $A$ is an $SD(\wedge)$ algebra
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The Prague Theorem

Theorem (BK)

Let $A$ be an algebra. TFAE

- $A$ is an SD($\land$) algebra
- Every Prague strategy over $A$ has a solution

Proof.

Implication $\uparrow$ follows from Larose, Zádori, Bulatov
The Prague Theorem

**Theorem (BK)**

Let $A$ be an algebra. TFAE

- $A$ is an SD($\land$) algebra
- Every Prague strategy over $A$ has a solution

**Proof.**

Implication $\uparrow$ follows from Larose, Zádori, Bulatov

For $\downarrow$ the strategy of the proof is to find smaller and smaller substrategies until we find a solution.
The Prague Theorem

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Two cases

- When we have a proper absorbing set of the projection to some singleton
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Two cases

- When we have a proper absorbing set of the projection to some singleton
- When we don’t have …
Thank you for your attention!

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