# Prague Strategies

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# The Prague Theorem

## Theorem (Barto, Kozik 2009)

*Let* **A** *be an idempotent algebra. TFAE*

- **A** *is an* $\mathrm{SD}(\wedge)$ *algebra*
  *(= lies in a variety omitting* $\mathbf{1}$ *and* $\mathbf{2}$*)*
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Plan:

- $k$-intersection property
- $\mathrm{SD}(\wedge)$
- $\mathrm{CSP}(\mathbf{A})$
- $(k, l)$-minimal instance
- Prague strategy

# Warning

All algebras are finite and idempotent

# $k$-intersection property

## Definition ($k$-equal relations)

*$R_1, R_2 \subseteq A^n$ are k-equal, if $\forall J \subseteq [n]$, $|J| \leq k$, the projections of $R_1$ and $R_2$ to $J$ are equal.*

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*A finite algebra $\mathbf{A}$ satisfies the k-intersection property, if $\forall n$ every collection of pairwise k-equal non-empty subuniverses $R_1, \ldots, R_m \leq \mathbf{A}^n$ has nonempty intersection.*

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## Observation

$\mathbf{B} \in \mathrm{HSP}(\mathbf{A})$. Then

$\mathbf{A}$ has the $k$-intersection property $\Rightarrow$ $\mathbf{B}$ has the $k$-intersection prop.

# $k$-IP, modules are bad

**Observation**

If $\mathbf{A}$ is a reduct of a module and $|A| > 1$, then $\mathbf{A}$ fails the $k$-intersection property for every $k$.

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### Proof.

For $a \in A$ let

$$R_a = \{(a_1, \dots, a_{k+1}) : a_1 + a_2 + \dots + a_{k+1} = a\}$$

Clearly

- $R_a$ is a subuniverse of $\mathbf{A}^{k+1}$
- any projection to less than $k+1$ coordinates is full
- if $a \neq b$ then $R_a \cap R_b = \emptyset$

□

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## Corollary

If $\mathbf{A}$ has the $k$-intersection property for some $k$, then $\mathsf{HSP}(\mathbf{A})$ doesn't contain a reduct of a module (with more than one element).

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**Conjecture (Valeriote)**

The other implication is also true.

# SD($\wedge$)

**Theorem (Hobby, Maróti, McKenzie, Valeriote, Willard)**

*Let* $\mathbf{A}$ *be an algebra. TFAE*

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  *if* $\mathbf{B} \in \mathrm{HSP}(\mathbf{A})$, $\alpha, \beta_1, \beta_2 \in \mathrm{Con}(\mathbf{B})$
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## Definition

**A** *is* $\mathrm{SD}(\wedge)$*, if it satisfies the equivalent conditions above*

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- If **A** has a short (3-terms) chain of Jónsson terms, then **A** has the 2-intersection property Kiss, Valeriote and 2 is the optimal number

# CSP

## Definition ($\mathrm{CSP}(\mathbf{A})$)

*Let* $\mathbf{A}$ *be an algebra. An instance of* $\mathrm{CSP}(\mathbf{A})$ *is a pair* $(V, \mathcal{C})$*, where*

- $V$ *is a finite set (elements are called variables)*
- $\mathcal{C}$ *is a finite set of constraints*

  *Constraint is a subuniverse* $C$ *of* $\mathbf{A}^D$*, where*

  $D \subseteq V$ *(called the scope of* $C$*)*

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*A solution to* $(V, \mathcal{C})$ *is a mapping* $f : V \to A$ *which satisfies all the constraints* $C \leq \mathbf{A}^D$ *in* $\mathcal{C}$*, i.e.* $f|D \in C$.

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The aim is to find a solution fast (in poly-time).

# $(k, l)$-minimal instance

## Definition ($\pm$ Bulatov, Jeavons)

*Let $k \leq l$ be natural numbers.*
*An instance $(V, \mathcal{C})$ of $\mathrm{CSP}(\mathbf{A})$ is called $(k, l)$-minimal if*

- *Every $l$-element subset of $V$ is a subset of the scope of some constraint in $\mathcal{C}$*
- *For every $J \subseteq V$, $|J| \leq k$ and every pair $C_1, C_2 \in \mathcal{S}$ whose scopes contain $J$, the projections of $C_1$ and $C_2$ onto $J$ are the same.*

*An instance $(V, \mathcal{C})$ is called $k$-minimal, if it is $(k, k)$-minimal.*

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**Observation**

If $k' \leq k$ and $l' \leq l$ then $(k, l)$-minimal instance is $(k', l')$-minimal.

# Bounded relational width

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Every instance of $\mathrm{CSP}(\mathbf{A})$ can be converted into an equivalent $(k, l)$-minimal instance in poly-time.

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## Definition

**A** *has relational width $(k, l)$ if every $(k, l)$-minimal instance, whose constraints are non-empty, has a solution.*

**A** *has bounded relational width if it has relational width $(k, l)$ for some $k, l$.*

# The bounded relational width conjecture

**Theorem (Larose, Zádori, Bulatov)**

*If* **A** *has bounded relational width, then* **A** *is an* $\mathrm{SD}(\wedge)$ *algebra.*

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- If **A** has a short chain of Jónsson terms (4 terms), then **A** has "bounded width" Carvalho, Dalmau, Marković, Maróti

# Corollaries of the Prague Theorem

## Corollary

If **A** is an $\mathrm{SD}(\wedge)$ algebra, then **A** has relational width $(2, 3)$.
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If $\mathbf{A}$ is an $\mathrm{SD}(\wedge)$ algebra, then $\mathbf{A}$ satisfies the 2-intersection property. (Recall that 2 is optimal.)

## Proof.

- Let $R_1, \ldots, R_m \leq \mathbf{A}^n$ be nonempty and 2-equal
- Let $V = [n]$, $\mathcal{C} = \{R_1, \ldots, R_m\}$
- $(V, \mathcal{C})$ is a $(2, n)$-minimal instance of $\mathrm{CSP}(\mathbf{A})$

□

# No idea about the title

I am finally going to introduce Prague strategies.

Comparison with known notions:

$$(2,3)\text{-minimal instance of } \mathrm{CSP}(\mathbf{A})$$
$$\Downarrow$$
$$\text{Prague strategy over } \mathbf{A}$$
$$\Downarrow$$
$$1\text{-minimal instance of } \mathrm{CSP}(\mathbf{A})$$

# Patterns

Let $(V, \mathcal{C})$ be an instance of $\mathrm{CSP}(\mathbf{A})$

# Patterns

For $x, y \in V$ and $C \in \mathcal{C}$ and $a, b \in A$ we write $a \xrightarrow{x,y,C} b$, if

- $x, y$ are in the scope of $C$
- The mapping $x \rightarrow a, y \rightarrow b$ is in the projection of $C$ to $\{x, y\}$

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## Definition

*A* *pattern* $w$ *is a tuple* $(x_1, C_1, \dots)$:

$$x_1 \xrightarrow{C_1} x_2 \xrightarrow{C_2} \dots \xrightarrow{C_i} x_{i+1},$$

*where* $x_j \in V$ *and* $C_j \in \mathcal{C}$.

*We write* $a \xrightarrow{w} b$, *if there exist* $a = a_1, a_2, \dots, a_{i+1} = b$ *such that*

$$a = a_1 \xrightarrow{x_1,x_2,C_1} a_2 \xrightarrow{x_2,x_3,C_2} a_3 \to \cdots \to a_i \xrightarrow{x_i,x_{i+1},C_i} a_{i+1} = b$$

*The* *scope* *of* $w$ *is* $[[w]] = \{x_1, \dots, x_{i+1}\}$

# Prague strategy

If patterns $w_1, w_2$ start and end with the same variable $x$, we can form their concatenation $w_1 \circ w_2$.

$w^K = w \circ w \circ \cdots \circ w$ ($K$-times)

# Prague strategy

## Definition (!!!!!!!)

*A Prague strategy over* $\mathbf{A}$ *is an instance* $(V, \mathcal{C})$ *of* $\mathrm{CSP}(\mathbf{A})$ *such that*

- $(V, \mathcal{C})$ *is* 1*-minimal*
- *For every* $x \in V$,
  *every pattern* $v$ *starting and ending with* $x$,
  *every* $a, b \in A$ *such that* $a \xrightarrow{v} b$ *and*
  *every pattern* $w$ *starting and ending with* $x$ *s.t.* $[[v]] \subseteq [[w]]$,

  *there exists a natural number* $K$ *such that* $a \xrightarrow{w^K} b$

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## Observation

Every $(2, 3)$-min. instance of $\mathrm{CSP}(\mathbf{A})$ is a Prague strategy over $\mathbf{A}$.

# The Prague Theorem

## Theorem (BK)

*Let* **A** *be an algebra. TFAE*

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Implication $\Uparrow$ follows from Larose, Zádori, Bulatov

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Two cases

- When we have a proper absorbing set of the projection to some singleton

□

# The Prague Theorem

## Theorem (BK)

*Let* **A** *be an algebra. TFAE*

- **A** *is an* $\mathrm{SD}(\wedge)$ *algebra*
- *Every Prague strategy over* **A** *has a solution*

## Proof.

Implication $\Uparrow$ follows from Larose, Zádori, Bulatov

For $\Downarrow$ the strategy of the proof is to find smaller and smaller substrategies until we find a solution

Two cases

- When we have a proper absorbing set of the projection to some singleton
- When we don't have ...

$\square$

# Lunch!

thAnk yoU FOr youR ATtentiON!

ThANK you fOR your atTENTion?

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