Cyclic terms for join semi-distributive varieties II

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joint work with Libor Barto

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Kraków, Poland

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Cyclic algebras

Definition (Cyclic algebra)

Let $A$ be a finite algebra. A subalgebra $R \leq A^n$ is cyclic, if

$$\forall a_1, \ldots, a_n \in A \quad (a_1, a_2, \ldots, a_n) \in R \Rightarrow (a_2, \ldots, a_n, a_1) \in R$$
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- $R$ is a relation on $A$ compatible with operations of algebra $A$
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Definition (A retraction)

For a relational structure $(A, R)$ a function $f : A \rightarrow A$ is a retraction iff

- $f(f(a)) = f(a)$ for all $a \in A$ and
- if $(a_1, \ldots, a_n) \in R$ then $(f(a_1), \ldots, f(a_n)) \in R$ (endomorphism).
The missing theorem

Theorem (For simple algebras)

Let $A$ be a finite, simple algebra from an $SD(\vee)$ variety and

and more generally

Theorem (General case)

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**Theorem (For simple algebras)**

Let $A$ be a finite, simple algebra from an $SD(\vee)$ variety and let $p$ be a prime number greater than $|A|$ and let $R \leq A^p$ be a cyclic and subdirect subalgebra of $A^p$. If $R$ has more than one element then $(A, R)$ has a non-trivial retraction.

And more generally

**Theorem (General case)**

Let $A$ be a finite, simple algebra from an $SD(\vee)$ variety and let $p$ be a prime number greater than $|A|$ and let $R \leq A^p$ be a cyclic and subdirect subalgebra of $A^p$. Then $R$ contains a constant tuple.
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The general case from the simple case

Let $A$ and $R$ be the minimal counterexample to the general case. And, WLOG, the algebras are idempotent.
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  - and $f(R) = (R \cap B^p) \leq B^p$ is a cyclic subalgebra of $B$
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  - if $t_1, \ldots, t_m$ are idempotent operations of $A$ satisfying a Maltsev condition for $SD(\lor)$;
  - then, putting $B = (f(A), ft_1, \ldots, ft_m)$,
  - $B$ is an $SD(\lor)$ algebra (as $ft_1, \ldots, ft_m$ satisfy all linear equations satisfied by $t_1, \ldots, t_m$)
  - and $f(R) = (R \cap B^p) \leq B^p$ is a cyclic subalgebra of $B$
  - from minimality of $A$ it contains a constant tuple ... OK
The simple case

Let $A$ be an $SD(\lor)$ algebra and $R \leq A^p$ be cyclic (for a prime $p > |A|$).
The simple case

Let $A$ be an SD($\lor$) algebra and $R \leq A^p$ be cyclic (for a prime $p > |A|$).

Definition

An unfolding of $R$ is the $p$-ary relation $R'$ on the set $A \times p$ defined by

$$R' = \left\{ ((a_1, 1), \ldots, (a_p, p)) \in (A \times p)^p \mid (a_1, \ldots, a_p) \in R \right\}.$$
The simple case

Let $A$ be an SD($\lor$) algebra and $R \leq A^p$ be cyclic (for a prime $p > |A|$).

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An unfolding power $C$ of $R$ is the subset of $A^{A \times p}$ consisting of all the homomorphisms from the relational structure $(A \times p, R')$ to $(A, R)$. 
The simple case

Let $\mathbf{A}$ be an $\text{SD}(\vee)$ algebra and $\mathbf{R} \leq \mathbf{A}^p$ be cyclic (for a prime $p > |\mathbf{A}|$).

**Definition**

An unfolding of $\mathbf{R}$ is the $p$-ary relation $\mathbf{R}'$ on the set $\mathbf{A} \times p$ defined by

$$\mathbf{R}' = \{((a_1, 1), \ldots, (a_p, p)) \in (\mathbf{A} \times p)^p \mid (a_1, \ldots, a_p) \in \mathbf{R}\}.$$

An unfolding power $\mathbf{C}$ of $\mathbf{R}$ is the subset of $\mathbf{A}^{\mathbf{A} \times p}$ consisting of all the homomorphisms from the relational structure $(\mathbf{A} \times p, \mathbf{R}')$ to $(\mathbf{A}, \mathbf{R})$.

- $\mathbf{C}$ is a subuniverse of $\mathbf{A}^{\mathbf{A} \times p}$
The simple case

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An unfolding power $C$ of $R$ is the subset of $A^{A \times p}$ consisting of all the homomorphisms from the relational structure $(A \times p, R')$ to $(A, R)$.

- $C$ is a subuniverse of $A^{A \times p}$
- any $g \in C$ is a tuple $(g_1, \ldots, g_p)$ (where $g_i(a) = g((a, i))$) and

  $$(g_1(a_1), \ldots, g_p(a_p)) \in R \text{ whenever } (a_1, \ldots, a_p) \in R.$$
The simple case

Let $A$ be an $SD(\lor)$ algebra and $R \leq A^p$ be cyclic (for a prime $p > |A|$).

**Definition**

An *unfolding* of $R$ is the $p$-ary relation $R'$ on the set $A \times p$ defined by

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- $(\text{id}_A, \text{id}_A, \ldots, \text{id}_A) \in C$
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Let \( A \) be an SD(\( \vee \)) algebra and \( R \leq A^p \) be cyclic (for a prime \( p > |A| \)).

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An **unfolding** of \( R \) is the \( p \)-ary relation \( R' \) on the set \( A \times p \) defined by

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An **unfolding power** \( C \) of \( R \) is the subset of \( A^{A \times p} \) consisting of all the homomorphisms from the relational structure \((A \times p, R')\) to \((A, R)\).

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- \( (\text{id}_A, \text{id}_A, \ldots, \text{id}_A) \in C \)
- \( (\overline{a_1}, \ldots, \overline{a_p}) \in C \) for \( (a_1, \ldots, a_p) \in R \),
Let $A$ be an SD($\lor$) algebra and $R \leq A^p$ be cyclic (for a prime $p > |A|$).

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- if \((g, g, \ldots, g) \in C \) then \( g \) is an endomorphism of \((A, R)\);
- if \((f_1, \ldots, f_p), d = (g_1, \ldots, g_p) \in C \), then \((f_1 \circ g_1, \ldots, f_p \circ g_p) \in C \).
The simple case reloaded

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- if $(g, g, \ldots, g) \in C$ then $g$ is an endomorphism of $(A, R)$;
- if $(f_1, \ldots, f_p)$, $d = (g_1, \ldots, g_p) \in C$, then $(f_1 \circ g_1, \ldots, f_p \circ g_p) \in C$;
- if $(g_1, \ldots, g_p) \in C$ then $(g_j, g_{j+1}, \ldots, g_{j+p}) \in C$ for any $j$. 
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- if $(g_1, \ldots, g_p) \in C$ then $(g_j, g_{j+1}, \ldots, g_{j+p}) \in C$ for any $j$.  

The simple case concluded

**Definition**

For two tuples \((f_1, \ldots, f_p), (g_1, \ldots, g_p) \in C\) we define a congruence \(\eta_j\)

\[(f_1, \ldots, f_p) \eta_j (g_1, \ldots, g_p) \text{ iff } (f_i = g_i \text{ for all } i \neq j)\]
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From the SD(∨) we infer that

\[\bigvee_j \eta_j\] is the full congruence on \(C\)
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and therefore there is \(j\), and not onto \(f_j : A \to A\) such that

\[(\text{id}_A, \text{id}_A, \ldots, \text{id}_A) \eta_j (f_1, \ldots, f_p)\]
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and therefore there is \(j\), and not onto \(f_j : A \to A\) such that

\[(\text{id}_A, \text{id}_A, \ldots, \text{id}_A) \eta_j (f_1, \ldots, f_p)\]

thus \(f_i = \text{id}_A\) for \(i \neq j\) and finally

\[(f_j, \text{id}_A, \ldots, \text{id}_A) \circ (\text{id}_A, f_j, \text{id}_A, \ldots, \text{id}_A) \circ \cdots \circ (\text{id}_A, \ldots, \text{id}_A, f_j) = (f_j, \ldots, f_j)\]
Thank you for your attention.