Příklad 4.10.* $\int_{-1}^{1} \frac{dx}{\sqrt{(1-2ax+a^2)(1-2bx+b^2)}}$, $|a|<1$, $|b|<1$, $ab>0$

pomocná fce $g(y) := \frac{1+y^2}{2y}$, $y \in \langle -1,0) \cup (0,1\rangle$

$$g'(y) = \frac{2y\cdot 2y-(1+y^2)2}{4y^2} = \frac{y^2-1}{2y^2} < 0 \quad \text{pro } y \in (-1,0)\cup(0,1)$$

$\Rightarrow$ g je klesající na intervalech $(-1,0)$ a $(0,1)$

g nabývá maxima na $\langle -1,0)$ v bodě $y=-1$, $g(-1)=-1$

g nabývá minima na $(0,1\rangle$ v bodě $y=1$, $g(1)=1$

obor hodnot: $\mathcal{H}(g) = (-\infty,-1\rangle \cup \langle 1,+\infty)$

$P(x) := (1-2ax+a^2)(1-2bx+b^2)$ má kořeny $x_1 = \frac{1+a^2}{2a}$ a $x_2 = \frac{1+b^2}{2b}$

$|a|<1$, $|b|<1$ $\Rightarrow$ $x_1, x_2 \in (-\infty,-1)\cup(1,+\infty)$

$a>0$, $b>0$ $\Rightarrow$ $x_1>1$, $x_2>1$

$a<0$, $b<0$ $\Rightarrow$ $x_1<-1$, $x_2<-1$

v obou případech $1-2ax+a^2>0$ a $1-2bx+b^2>0$ pro $x\in(-1,1)$

$\Rightarrow$ $f(x) := \frac{1}{\sqrt{(1+a^2-2ax)(1+b^2-2bx)}}$ def. a spoj. na $(-1,1)$

I. $b=a$

$$\int_{-1}^{1} \frac{dx}{\sqrt{(1+a^2-2ax)^2}} = \int_{-1}^{1} \frac{dx}{1+a^2-2ax} = -\frac{1}{2a}\int_{-1}^{1} \frac{dx}{x-\frac{1+a^2}{2a}} = -\frac{1}{2a}\left[\log\left|x-\frac{1+a^2}{2a}\right|\right]_{-1}^{1}$$

$$= -\frac{1}{2a}\log\left|\frac{1-\frac{1+a^2}{2a}}{-1-\frac{1+a^2}{2a}}\right| = -\frac{1}{2a}\log\frac{(a-1)^2}{(a+1)^2} \underset{\boxed{-1<a<1}}{=} -\frac{1}{a}\log\frac{1-a}{1+a}$$

II. $b\neq a$

substituce $t=\sqrt{\frac{(1+b^2)-2bx}{(1+a^2)-2ax}}$, $x\in(-1,1)$

$$x = \frac{(1+b^2)-(1+a^2)t^2}{2b-2at^2} =: \varphi(t)$$

$$\varphi'(t) = \frac{-2(1+a^2)t(2b-2at^2)+\left[(1+b^2)-(1+a^2)t^2\right]4at}{4(b-at^2)^2} = \frac{a(1+b^2)-b(1+a^2)}{(b-at^2)^2}\,t$$

$a(1+b^2)-b(1+a^2)>0 \iff \frac{1+b^2}{b} > \frac{1+a^2}{a} \iff b<a$

$\varphi'(t)>0$ pokud $b<a$ a $\varphi'(t)>0$ pokud $b>a$

$$\sqrt{(1+a^2-2at)(1+b^2-2bt)} = t(1+a^2-2ax) = t\left(1+a^2-\frac{2a(1+b^2)+2a(1+a^2)t^2}{2b-2at^2}\right)$$

$$= t\,\frac{b(1+a^2)-a(1+b^2)}{b-at^2}$$

Meze: BÚNO $b<a$, pak φ je rostoucí

$$x=-1: \quad t=\sqrt{\frac{(1+b)^2}{(1+a)^2}}=\frac{1+b}{1+a}$$

$$x=1: \quad t=\sqrt{\frac{(1-b)^2}{(1-a)^2}}=\frac{1-b}{1-a}$$

$$\int_{-1}^{1}\frac{dx}{\sqrt{(1+a^2-2ax)(1+b^2-2bx)}} \overset{\text{VIII.24}}{=} \int_{\frac{1+b}{1+a}}^{\frac{1-b}{1-a}} \frac{1}{\cancel{t}\,\frac{b(1+a^2)-a(1+b^2)}{b-at^2}}\cdot\frac{a(1+b^2)-b(1+a^2)}{(b-at^2)^2}\,\cancel{t}\,dt =$$

$$= \int_{\frac{1+b}{1+a}}^{\frac{1-b}{1-a}} \frac{-1}{b-at^2}\,dt = \int_{\frac{1-b}{1-a}}^{\frac{1+b}{1+a}} \frac{1}{b-at^2}\,dt = \frac{1}{b}\int_{\frac{1-b}{1-a}}^{\frac{1+b}{1+a}} \frac{1}{1-\underbrace{\frac{a}{b}}_{>0}t^2}\,dt =$$

$$= \frac{1}{2b}\int_{\frac{1-b}{1-a}}^{\frac{1+b}{1+a}} \frac{1}{1-\sqrt{\frac{a}{b}}\,t}+\frac{1}{1+\sqrt{\frac{a}{b}}\,t}\,dt = \frac{1}{2\sqrt{ab}}\left[\log\left|\frac{1+\sqrt{\frac{a}{b}}\,t}{1-\sqrt{\frac{a}{b}}\,t}\right|\right]_{\frac{1-b}{1-a}}^{\frac{1+b}{1+a}} =$$

$$= \frac{1}{2\sqrt{ab}}\log\left|\frac{\left(1+\sqrt{\frac{a}{b}}\frac{1+b}{1+a}\right)\left(1-\sqrt{\frac{a}{b}}\frac{1-b}{1-a}\right)}{\left(1-\sqrt{\frac{a}{b}}\frac{1+b}{1+a}\right)\left(1+\sqrt{\frac{a}{b}}\frac{1-b}{1-a}\right)}\right| = \frac{1}{2\sqrt{ab}}\log\left|\frac{1+\sqrt{\frac{a}{b}}\left(\frac{1+b}{1+a}-\frac{1-b}{1-a}\right)-\frac{a}{b}\frac{1-b^2}{1-a^2}}{1-\sqrt{\frac{a}{b}}\left(\frac{1+b}{1+a}-\frac{1-b}{1-a}\right)-\frac{a}{b}\frac{1-b^2}{1-a^2}}\right| =$$

$$= \frac{1}{2\sqrt{ab}}\log\left|\frac{1+2\sqrt{\frac{a}{b}}(b-a)-\frac{a}{b}\frac{1-b^2}{1-a^2}}{1-2\sqrt{\frac{a}{b}}(b-a)-\frac{a}{b}\frac{1-b^2}{1-a^2}}\right| = \frac{1}{2\sqrt{ab}}\log\left|\frac{b(1-a^2)+2\sqrt{ab}(b-a)-a(1-b^2)}{b(1-a^2)-2\sqrt{ab}(b-a)-a(1-b^2)}\right| =$$

$$= \frac{1}{2\sqrt{ab}}\log\left|\frac{1+2\sqrt{ab}+ab}{1-2\sqrt{ab}+ab}\right| = \frac{1}{2\sqrt{ab}}\log\frac{(1+\sqrt{ab})^2}{(1-\sqrt{ab})^2} = \frac{1}{\sqrt{ab}}\log\left(\frac{1+\sqrt{ab}}{1-\sqrt{ab}}\right)$$