ON GROUP MODULES

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Abstract. The paper is focused on questions when some homological and submodule-chain conditions satisfied by a module M are preserved by the group module MG. Namely, it is proved for a group G and an R-module M that MG_{RG} is flat if and only if M_R is flat, and MG_{RG} is artinian if and only if M_R is artinian and G is finite, which are two questions raised by Yiqiang Zhou: On Modules Over Group Rings, Noncommutative Rings and Their Applications LENS July 1-4, 2013.

Throughout the paper R will always denote a ring with identity and the notion of an R-module will mean a unitary right module. Let us start with the key definition of a group module which generalizes the widely studied notion of a group ring. Suppose that G is a group, and M is a module over a ring R. Let MG denote the set all formal linear combinations of the form $\sum_{g\in G} m_g g$, where $m_g \in M$ and $m_g = 0$ for almost all g. Denote by RG the corresponding group ring and determine on MG structure of a right RG-module:

$$\sum_{g\in G} m_g g + \sum_{g\in G} n_g g = \sum_{g\in G}(m_g + n_g)g,$$

$$(\sum_{g\in G} m_g g)(\sum_{g\in G} h_g g) = \sum_{g\in G}(\sum_{h,h':hh'=g} m_h r'_h)g$$

for all elements $\sum_{g\in G} m_g g$, $\sum_{g\in G} n_g g \in MG$ and $\sum_{g\in G} r_g g \in RG$. Then the module structure MG_{RG} is correctly defined and it is said to be a *group module* over the group G by [5]. If we identify every element $m \in M$ with $m \cdot 1 \in MG$, it is easy to see that M is an R-submodule of MG, where 1 denotes the identity element of G. By [7, Lemma 2.1], if MG is a group module, then $MG \cong_{RG} M \otimes_R RG$.

In [13], Zhou asked the following two questions in his presentation:
Q1. Characterize when MG_{RG} is flat.
Q2. Characterize when MG_{RG} is artinian.

Let G be a group and M be a nonzero R-module. In this note, we answer these two questions:
• M_R is a flat R-module if and only if MG_{RG} is a flat RG-module (see Theorem 8).
• MG_{RG} is artinian if and only if M_R is artinian and G is finite (see Theorem 19).

Furthermore, we prove several necessary conditions of a group under which the group module satisfies some other conditions on chain of submodules, in particular:
• If MG_{RG} is semiartinian, then M_R is semiartinian (see Theorem 11).
• If MG_{RG} is noetherian, then both M_R and G are noetherian (see Theorem 20).

Throughout this article, for a submodule N of M, we use $N \leq M$ ($N < M$) to mean that N is a submodule of M (respectively, a proper submodule), and we write $N \leq^e M$ to indicate that N is an essential submodule of M. We write $J(R), J(M), \mathrm{Soc}(R), \mathrm{Soc}(M), Z(R)$ for the Jacobson radical of the ring R, for the radical of the module M, the socle of R, the socle of M and the singular ideal of R, respectively. For an element m of a module M, $r_R(m) = \{r \in R|\ mr = 0\}$ is the annihilator of m.

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1. Flat and ADS-modules

Recall that a right module M over a ring R is said to be *ADS* if for every decomposition $M = A \oplus B$ and every complement C of A, we have $M = A \oplus C$ ([1], see also, [6]).

Before we start to the investigate group ADS-modules, we need to recall the notion of an excellent extension, introduced by Passman [11], and named by Bonami [2].

Let R and S be rings with the same unity such that R is a subring of S. The ring S is an *excellent extension* of R if the following conditions are satisfied:

(1) If M is an S-module with an S-submodule ${}_SN$ and N is a direct summand of M as an R-module, then N is a direct summand of M as an S-module.

(2) There is a finite set $\{1 = s_1, s_2, \dots, s_n\} \subseteq S$ such that S is a free left and right R-module with a basis $\{1 = s_1, s_2, \dots, s_n\}$ and $Rs_i = s_iR$ for all $i = 1, \dots, n$.

As it is shown in [9], examples of excellent extensions include $n \times n$ matrix rings $M_n(R)$ and crossed product $R * G$, where G is a finite group with $|G|^{-1} \in R$.

We will need the following facts.

Lemma 1. [1, Lemma 3.1] *An R-module M is ADS if and only if for each decomposition $M = A \oplus B$, A and B are mutually injective.*

Lemma 2. [10, Corollary 1.4] *Let S be an excellent extension of R, and M and N be S-modules. If N_R is M_R-injective then N_S is M_S-injective.*

Lemma 3. [10, Lemma 1.5] *Let S be an excellent extension of R, and M and N be R-modules. If $N \otimes_R S_S$ is $M \otimes_R S_S$-injective then N_R is M_R-injective.*

Let us prove several elementary facts on submodules of a group module (cf. [5, 7]) For an R-module M, $\mathcal{S}_R(M)$ denotes the set of all submodules of M.

Lemma 4. *Let M be a nonzero R-module, G a group and H a subgroup of G.*

1. *The functor $- \otimes_{RH} RG : RH - Mod \to RG - Mod$ is exact, preserves direct limits, and $A \otimes_{RH} RG \neq 0$ for each nonzero RH-module A.*
2. *There exists the unique isomorphism $\varphi_M^H : MH \otimes_{RH} RG \to MG$ satisfying the condition $\varphi_M^H(mh \otimes g) = mhg$ for every $m \in M$, $h \in H$ and $g \in G$.*
3. *The map $\Phi_M^H : \mathcal{S}_{RH}(MH) \to \mathcal{S}_{RG}(MG)$ defined by the rule $\Phi_M^H(A) = \varphi_M^H(A \otimes_{RH} RG)$, where $A \otimes_{RH} RG$ is identified with the corresponding submodule of $MH \otimes_{RH} RG$, is injective and monotonic with respect to ordering by inclusion.*

Proof. (1) Let T be a right transversal of the subgroup H. Then $RG \cong_{RH} \bigoplus_{t\in T} RHt \cong_{RH} RH^{(|T|)}$ is a free left RH-module. Hence $- \otimes_{RH} RG$ is exact and

$$A \otimes_{RH} RG \cong_{RH} A \otimes_{RH} RH^{(|T|)} \cong_{RH} A^{(|T|)} \neq 0$$

for any $A \neq 0$. Moreover, the tensor functor $- \otimes_R RG$ preserves direct limits by Eilenberg-Watts Theorem because RG is obviously a flat R-module,.

(2) The existence of the surjective homomorphism φ_M^H follows from the universal property of the tensor product. The proof of the injectivity of φ_M^H is an easy exercise.

(3) First note that for $A \leq B \leq MH$ we have that $A \otimes_{RH} RG \leq B \otimes_{RH} RG \leq MH \otimes_{RH} RG$ (recall that we can identify the tensor product $N \otimes_{RH} RG$ for every $N \leq MH$ with a submodule $\{\sum_i n_i \otimes \alpha_i |\ n_i \in N, \alpha_i \in RH\}$ of the RG-module $MH \otimes_{RH} RG$, and in the sequel which follows by (1)). Since φ_M^H is an isomorphism, we obtain

$$\Phi_M^H(A) = \varphi_M^H(A \otimes_{RH} RG) \leq \varphi_M^H B \otimes_{RH} RG) = \Phi_M^H(B).$$

It remains to prove that Φ_M^H is injective. Let $A \neq B$. Then either $A \subsetneq A + B$ or $B \subsetneq A + B$. Without lost of the generality, we suppose the strictness of the first inclusion.

Applying the functor $- \otimes_{RH} RG$ on the exact sequence

$$0 \to A \to A + B \to B + A/A \to 0$$

we get by (1) the exact sequence

$$0 \to A \otimes_{RH} RG \overset{\alpha}{\to} (A + B) \otimes_{RH} RG \to (B + A/A) \otimes_{RH} RG \to 0.$$

Since the monomorphism α (identified with the inclusion) is not epimorphism, we have

$$A\otimes_{RH} \subsetneq (A+B)\otimes_{RH} RG = A\otimes_{RH} RG + B\otimes_{RH} RG,$$

which proves that $A\otimes_{RH} RG \neq B\otimes_{RH} RG$. As φ_M^H is an isomorphism,

$$\Phi_M^H(A) = \varphi_M^H(A\otimes_{RH} RG) \neq \varphi_M^H B\otimes_{RH} RG) = \Phi_M^H(B)$$

as desired. □

Proposition 5. *Let S be an excellent extension of R and let M be a right S-module.*

(1) *If M_R is an ADS-module, then so is M_S.*
(2) *If $M\otimes_R S_S$ is an ADS-module, then so is M_R.*

Proof. (1) Let $M = A_S \oplus B_S$. Then A_R is B_R-injective by Lemma 1 hence A_S is B_S-injective by Lemma 2, which suffices to prove by Lemma 1.

(2) Let $M = A_R \oplus B_R$. Then $M\otimes_R S_S = (A_R\otimes_R S_S)\oplus(B_R\otimes_R S_S)$ hence $A\otimes_R S_S$ is $A\otimes_R S_S$-injective by Lemma 1 and so A_R is B_R-injective by Lemma 3. Now it remains to use Lemma 1 again. □

Corollary 6. *Let M be an R-module and G be a finite group with an invertible order in R. If MG is an ADS RG-module, then M_R is an ADS R-module.*

Proof. Since $MG_{RG} \cong M\otimes_R RG_{RG}$ by Lemma 4(1) and RG is an excellent extension by [12, Lemma 1.1], we can apply Proposition 5(2). □

As Lemma 3 could be easy generalized for extensions which are excellent relatively to a module M, i.e. such that the second axiom holds only for direct summands of M, we could suppose invertibility of the group order in the ring $\mathrm{End}(M)$ instead of R.

However the notion of the ADS module naturally generalizes semisimple modules, [5, Theorem 2.3] cannot be directly generalized for an ADS module as:

Example 7. (1) Let F be a field and G an infinite cyclic group. Then $FG \cong F[x, x^{-1}]$ is a trivial ADS FG-module since it is a domain, however G is infinite.

(2) Let p be a prime and $\mathbb{Z}_p$ be a field and G a group both of order p. Then

$$\mathbb{Z}_pG \cong \mathbb{Z}_p[x]/(x^p-1) = Z_p[x]/(x-1)^p$$

is a local ring. Then $\mathbb{Z}_pG$ is an indecomposable $\mathbb{Z}_pG$-module, so it is ADS. However, the order of G is zero in $\mathbb{Z}_p$

Now, we characterize flat group modules.

Theorem 8. *Let G be a group and M an R-module. Then M is a flat R-module if and only if MG is a flat RG-module.*

Proof. ($\Rightarrow$) By [8, Theorem 4.34], the module M_R is a direct limit of a directed system $(F_i, i\in I)$ consisting of finitely generated free modules. Since $-\otimes_R RG$ preserves direct limits by Lemma 4(1), $MG_{RG} \cong M\otimes_R RG_{RG}$ is a direct limit of the directed system $(F_i\otimes_R RG, i\in I)$ consisting of free RG-modules, which is flat by [8, Proposition 4.4].

($\Leftarrow$) Applying [8, Theorem 4.34], we get that MG_{RG} is a direct limit of a directed system $(M_i, i\in I)$ consisting of finitely generated free RG-modules. Obviously M_i are free R-modules as well and $(M_i, i\in I)$ is a directed system in the category of R-modules. Then MG_R is a direct limit of free modules $(M_i, i\in I)$ in the category of R-modules by [4, Lemma 2.3]. Hence MG_R is flat by [8, Proposition 4.4]. Since M_R is a direct summand in MG_R, it is flat by [8, Proposition 4.2]. □

2. Modules satisfying some chain conditions

A module M is said to be *semiartinian* if every non-zero factor of M has a nonzero socle (or, equivalently, each non-zero factor of M contains a simple submodule). Given a semiartinian module M, the *socle chain* of M is a continuous strictly increasing chain $(M_\alpha | \alpha \leq \sigma)$ of submodules of M satisfying $M_{\alpha+1}/M_\alpha = \mathrm{Soc}(M/M_\alpha)$ for each $\alpha < \sigma$ and $M = M_\sigma$. Notice that every artinian module is semiartinian.

We start the section with an easy technical observation.

Lemma 9. *Let M be a nonzero R-module, $N \leq G$, $m \in M \setminus \{0\}$ and $m_1 \in M \setminus N$. If $Soc(M) = 0$ and $mR \cap N = 0$, then there exists $r \in R$ such that $mr \neq 0$ and $m_1 \notin mrR + N$.*

Proof. If $m_1 \notin mR + N$, then it suffices to take $r = 1$. Suppose that $m_1 \in mR + N$ and denote by π the canonical projection $M \to M/N$. Let us observe that $\mathrm{Soc}(mR) = 0$ because $\mathrm{Soc}(M) = 0$, and

$$\pi(m)R = \pi(mR) = mR + N/N \cong mR$$

as $mR \cap N = 0$. Since $\overline{0} \neq \pi(m_1) \in \pi(mR)$ and $\mathrm{Soc}(\pi(mR)) = 0$, there exists $\overline{P} \leq^e \pi(mR)$ such that $\pi(m_1) \notin \overline{P}$. This means that there exists $r \in R$ such that $\overline{0} \neq \pi(m)r \in \overline{P}$. Hence $mr \neq 0$ and $m_1 \notin mrR + N$. □

The following claim constitutes a basic step of our prove that semiartinian group modules have semiartinian underlying modules.

Lemma 10. *Let M be a nonzero R-module and G be a group. If $Soc(MG_{RG}) \neq 0$, then $Soc(M_R) \neq 0$.*

Proof. If $G = 1$, then $MG \cong M$ and there is nothing to prove. Let G be a nontrivial group and fix an element $m = \sum_{i=1}^n m_i g_i \in \mathrm{Soc}(MG)$ with a minimal n such that mRG is a simple RG-module. Note that m is non-zero and $r_R(m_a) = r_R(m_b) \neq R$ for all $a < b \leq n$, otherwise, if there is $s \in r_R(m_a) \setminus r_R(m_b)$, then $ms = \sum_{i=1, i\neq a}^n m_i s g_i$ gives an example of a shorter element generating the same simple module.

Assume to contrary that $\mathrm{Soc}(M_R) = 0$. We will show by the induction on t that for every $t = 0, \dots n$ there exists $s \in R \setminus r_R(m_1)$ such that $m_1 \notin \sum_{i=1}^t m_i sR$. Since $m_1 \neq 0$, the claim is clear for $s = 1$ and $t = 0$.

Suppose that there exists $s_{t-1} \in R \setminus r_R(m_1)$ such that $m_1 \notin \sum_{i=1}^{t-1} m_i s_{t-1} R$. Let us put $N = \sum_{i=1}^{t-1} m_i s_{t-1} R$ and we will prove the claim is true for t. If there exists $r \in R$ such that $0 \neq m_t s_{t-1} r \in N$ we put $s = s_{t-1} r$ and we are done because $r_R(m_1) = r_R(m_t)$. Otherwise suppose $m_t s_{t-1} R \cap N = 0$. As $m_1 s_{t-1} \neq 0$ and so $m_t s_{t-1} \neq 0$ we may apply Lemma 9, hence there exists $r \in R \setminus r_R(m_t s_{t-1})$ such that $m_1 \notin m_i s_{t-1} rR + N \supseteq m_i s_{t-1} rR + \sum_{i=1}^t m_i s_{t-1} rR$. If we put $s = s_{t-1} r$, then $m_1 \notin \sum_{i=1}^{t-1} m_i s_{t-1} R$. Since $r_R(m_1) = r_R(m_t)$, we can see $m_1 s \neq 0$, hence then proof of the induction step is done.

Let s be an element for which $m_1 s \neq 0$ and $m_1 \notin \sum_{i=1}^n m_i sR$. Then $0 \neq ms \in mRG$, so $msRG = mRG$ as mRG is simple. Hence there exists an element $\rho = \sum_j r_j h_j \in RG$ for which $ms\rho = m$. Thus $m_1 = \sum_{i,j: g_1 = g_i h_j} m_i s r_j$ which contradicts to $m_1 \notin \sum_{i=1}^n m_i sR$. □

Theorem 11. *Let M be an R-module and G be a group. If MG_{RG} is semiartinian then M_R is semiartinian.*

Proof. Let N be an arbitrary proper submodule of M. It is enough to show that $\mathrm{Soc}(M/N) \neq 0$. Since NG is a proper submodule of MG and a nonzero factor of a semiartinian module is semiartinian, we get that $MG/NG \cong (M/N)G$ has an essential socle. Hence $\mathrm{Soc}(M/N)$ is nonzero by Lemma 10. □

Note that Example 7(1) shows that for an infinite cyclic group G and a field F, the FG-module FG is not semiartinian however F is even artinian.

Using a result of the work [5] about semisimple group modules, we characterize semiartinian group modules over finite groups having invertible order in its endomorphism ring.

Proposition 12. *Let M be an R-module and G be a finite group with order invertible in $End_R(M)$. Then M_R is semiartinian if and only if MG_{RG} is semiartinian.*

Proof. Suppose that M_R is semiartinian with the socle chain $(M_\alpha | \alpha \leq \sigma)$. Since $M_{\alpha+1}/M_\alpha$ is a semisimple R-module, $M_{\alpha+1}G_{RG}/M_\alpha G_{RG} \cong (M_{\alpha+1}/M_\alpha)G_{RG}$ is a semisimple RG-module by [5, Theorem 3.2] for every $\alpha < \sigma$. Thus MG_{RG} is semiartinian.

If, on the other hand, MG_{RG} is a semiartinian RG-module, then M_R is a semiartinian R-module by Theorem 11. □

The following claim shows that several constructions of non-artinian group rings work also in the case of group modules.

Proposition 13. *Let M be a nonzero R-module and G be a group. If*

(1) *either G is an infinite cyclic group*
(2) *or G contains an infinite strictly increasing chain of finite subgroups,*

then MG_{RG} is not artinian.

Proof. (1) Let g be a generator of a cyclic group G and $m \in M \setminus \{0\}$. Define a cyclic submodule $M_n = m(1+g)^n RG$ for every n. Then $M_1 \supseteq M_2 \supseteq \dots$ forms a decreasing chain of submodules and it remains to prove that $M_n \supsetneq M_{n+1}$ for every n.

Assume that there exists n such that $M_n = M_{n+1}$. There are integers u, v and $\alpha = \sum_{i=u}^{v} a_i g^i \in RG$ such that $u \leq v$, $ma_u \neq 0 \neq ma_v$ and

$$m(1+g)^n = m(1+g)^{n+1} \sum_{i=u}^{v} a_i g^i = m(1+g)^n (a_u g^u + \sum_{i=u+1}^{v} (a_i + a_{i-1}) g^i + a_v g^{v+1}).$$

Comparing coefficients of g^u in case that $u < 0$ we obtain that $ma_u = 0$, a contradiction. If $u \geq 0$, then $v \geq u \geq 0$, and comparing coefficients of g^{n+v+1} we get equality $ma_v = 0$, which contradicts to chose of α.

Since $M_1 \supsetneq M_2 \supsetneq \dots$ is a strictly decreasing chain of submodules, MG is not an artinian RG-module.

(2) Let $H_1 \subsetneq H_2 \subsetneq \dots$ be a strictly increasing chain finite subgroups of G and $m \in M \setminus \{0\}$. Put $\gamma_i = \sum_{h \in H_i} mh$ for each i. If T is a right transversal of the subgroup H_i in the group H_{i+1}, then $\gamma_{i+1} = \gamma_i \cdot \sum_{t \in T} 1t$, which proves that $\gamma_{i+1} \in \gamma_i RG$. Furthermore, if $\sum_g m_g g \in \gamma_{i+1} RG$, then $m_1 = m_h$ for every $h \in H_{i+1}$. Since $H_i \subsetneq H_{i+1}$ we see that $\gamma_i \notin \gamma_{i+1} RG$. We have constructed a strictly decreasing chain of submodules $\gamma_1 RG \supsetneq \gamma_2 RG \supsetneq \dots$ which witnesses that MG is not artinian. □

Recall that a group G is called *locally finite* if every finitely generated subgroup of G is finite and G is *periodic* if all its elements have a finite order.

Example 14. (1) Let $G = \mathbb{Z}_{p^\infty}$ be a Prüfer p-group for a prime p. Then G is a periodic artinian group and MG is non-artinian for every nonzero artinian module M by Proposition 13(2).

(2) If G is an infinite locally finite group, it contains an infinite set $\{g_i | i \in \mathbb{N}\} \subseteq G$ such that $g_n \notin \langle g_1, \dots g_{n-1} \rangle$ for each n. Then $H_i = \langle g_1, \dots g_i \rangle$, $i \in \mathbb{N}$ forms an infinite strictly increasing chain of finite subgroups, so MG_{RG} is non-artinian by Proposition 13(2) for an arbitrary nonzero module M. In particular, if $G = \mathbb{Q}/\mathbb{Z}$, we can see that the structure of decreasing chains of submodules is very reach by Lemma 4.

(3) If G contains an infinite cyclic subgroup $\langle g \rangle$, then $M\langle g \rangle_{R\langle g \rangle}$ is non-artinian by Proposition 13(1), hence we can find a strictly decreasing chain of submodules in MG_{RG} by Lemma 4(3).

The following observation is a straightforward consequence of Lemma 4.

Lemma 15. *Let M be an R-module, G a group and H a subgroup of G. If MG is artinian (noetherian), then MH is artinian (noetherian) as well.*

Proof. If $M = 0$, there is nothing to prove. If $(A_i | \ i \in \mathbb{N})$ is a strictly decreasing (increasing) chain of submodules of MH, then we have $(\Phi_M^H(A_i) | \ i \in \mathbb{N})$ forms a strictly decreasing (increasing) chain of submodules of MG by Lemma 4(3). □

Proposition 16. *Let M be an R-module and G be a group.*

(1) *If M is artinian (noetherian) and G is finite, then MG_{RG} is artinian (noetherian).*
(2) *If MG_{RG} is artinian then M_R is artinian and G is periodic.*

Proof. (1) Since $MG \cong_R M^{(|G|)}$ is an artinian (noetherian) R-module, it is also artinian (noetherian) as an RG-module.

(2) Note that $M\langle g\rangle$ isan artinian $R\langle g\rangle$-module for each $g \in G$ by Lemma 15, in particular $M \cong_R M\langle 1\rangle$ is an artinian R-module. Since $M\langle g\rangle$ is artinian, the cyclic group $\langle g\rangle$ is finite by Proposition 13(1), which proves that G is periodic. □

It is well known that if $e \in R$ is an idempotent and M is an R-module, then e is identity of the unitary ring eRe and Me has a natural structure of eRe-module.

Lemma 17. *Let $e \in R$ be an idempotent, M a nonzero R-module and G a group.*

(1) *If Ke and Le are eRe-submodules of the module Me such that $K \subsetneq L$, then KeR and LeR are R-submodules of M and $KeR \subsetneq LeR$.*
(2) *If M is an artinian (noetherian) R-module, then Me is an artinian (noetherian) eRe-module.*
(3) *If MG is an artinian (noetherian) RG-module, then MeG is an artinian (noetherian) $eReG$-module.*

Proof. (1) As $K \subsetneq L$, we obtain that $KeR \subseteq LeR$ are submodules of the R-module M. Assume that $KeR = LeR$. Then $K = KeRe = LeRe = L$, which contradicts to the hypothesis $K \subsetneq L$.

(2) If $(N_i)_{i\in\mathbb{N}}$ is a strictly decreasing (increasing) chain of eRe-submodules of the module Me, then $(N_iR)_{i\in\mathbb{N}}$ forms a strictly decreasing (increasing) chain of R-submodules of M by (1).

(3) Since R is a subring of the group ring RG, the element e is an idempotent of RG. Furthermore e commutes with all elements of G, hence $MGe = MeG$ is a module over $eRGe = eReG$. Now the claim follows from (2). □

The key role in our main result presents the following translation of an artinian or noetherian group module over simple module to a construction of an artinian or noetherian group ring.

Proposition 18. *Let S be a simple R-module, G be a group and $T = End(S_R)$. Then T is a skew-field and*

(1) *if SG is an artinian RG-module, then TG is a right artinian ring,*
(2) *if SG is a noetherian RG-module, then TG is a right noetherian ring.*

Proof. Since S is simple, it is easy to see that T is a skew-field, hence ${}_TS \cong_T T^{(\kappa)}$ for some cardinal number κ has the structure of a free left T-module, i.e. of a vector space over the skew-field T. Put $A = \mathrm{End}({}_TS) \cong \mathrm{End}({}_TT^{(\kappa)})$. Then there exists an idempotent $e \in A$ such that $eAe \cong T$ (any endomorphism which performs as identity on some one-dimensional subspace and it is zero on some complements). Note that S has the structure of the A-module and R can be seen as a subring of A, so RG is also a subring of AG. Since S is a simple R-module, it is a simple A-module. Moreover, as SG is an artinian (noetherian) RG-module, it is an artinian (noetherian) AG-module. Now, by Lemma 17(3), SeG is an artinian (noetherian) $eAeG$-module. As $eAe \cong T$ and as Se is a simple module over eAe, we obtain that TG is an artinian (noetherian) TG module, which finishes the proof. □

The previous proposition allows us to translate celebrated Connels' results on chain conditions of group rings [3] to the case of group modules.

Theorem 19. *Let R be a ring, G a group, and M be a nonzero R-module. Then MG_{GR} is artinian if and only if M_R is artinian and G is finite.*

Proof. Note that the reverse implication follows immediately from Proposition 16(1). Suppose that MG_{GR} is artinian. Then M_R is artinian by Proposition 16(2), so it remains to prove that G is finite. Let $S \subseteq M$ be a simple submodule of M. Then SG is a submodule of MG, hence artinian module. Then $T = \mathrm{End}(S_R)$ is a skew-field for which TG is a right artinian ring by Proposition 18. Hence G is finite by [3, Theorem 1]. □

We say that a group G is noetherian if it satisfies ACC on subgroups.

Theorem 20. *Let R be a ring, G a group, and M a nonzero R-module. If MG_{GR} is noetherian, then both M_R and G are noetherian.*

Proof. The module M_R is noetherian by Proposition 16(1). Thus there exists a maximal submodule $N \leq M$ and $S = M/N$ is a simple R-module. As MG_{GR} is noetherian, the module $SG \cong MG/NG$ is noetherian as well. Applying Proposition 18 again we get that TG is a right noetherian group ring for a skew-field $T = \mathrm{End}(S_R)$. Now, the claim follows from [3, Theorem 2(b)]. □

We finish the paper by listing several corresponding open problems from which the formulation of the third one is due to Zhou [13] and the last one is for long time open even in context of group rings:

Question. Describe equivalent conditions on a module M and a group G under which MG is semiartinian, ADS, pure injective, or noetherian.

References

[1] A. Alahmadi, S. K. Jain, A. Leroy: ADS modules. J. Algebra 352(2012), 215-222.

[2] L. Bonami: On the structure of skew group rings. Algebra Berichte 43, Verlag Reinhard Fischer, Munchen, 1984.

[3] I.G. Connel: On the group ring, Can. J. Math. 15 (1963), 650-685.

[4] R. Göbel and J. Trlifaj: Approximations and Endomorphism Algebras of Modules, 2nd edition, Walter de Gruyter, Berlin, 2012.

[5] M. T. Koşan, T. K. Lee and Y. Zhou: On modules over group rings, Algebras and Representation Theory, 17 (1)(2014) 87-102.

[6] M. T. Koşan and T. C. Quynh: ADS-modules and rings, Commun. Algebra, 42(8)(2014), 3541-3551.

[7] M. T. Koşan and J. Žemlička: Mod-retractable rings, Commun. Algebra, 42 (3)(2014), 998-1010.

[8] T.Y. Lam: Lectures on Modules and Rings, Springer-Verlag, New York, 1999.

[9] M. M. Parmenter and P. N. Stewart: Excellent extensions, Commun. Algebra, 16 (1988), 703-713.

[10] M.M. Parmenter and Y. Zhou: Relative injectivity of modules and excellent extensions, Quaes. Math. 22 (1) (1999) 101-107.

[11] D. S. Passman: The Algebraic Structure of Group Rings. New York: Wiley-Interscience, 1977.

[12] D. S. Passman: It's essentially Maschke's theorem. Rocky Mountain J. Math. 13 (1983), no. 1, 37-54.

[13] Y. Zhou: On Modules over Group Rings, conference presentation, Noncommutative Rings and Their Applications LENS, July 1-4, 2013.

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