UNIT AND IDEMPOTENT ADDITIVE MAPS OVER COUNTABLE LINEAR TRANSFORMATIONS

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Abstract. Let V be a countably generated right vector space over a field F and $\sigma \in End(V_F)$ be a shift operator. We show that there exist a unit u and an idempotent e such that $1-u, \sigma-u$ are units in $End(V_F)$ and $1-e, \sigma-e$ are idempotents in $End(V_F)$. We also obtain that if D is a division ring and $D \ncong \mathbb{Z}_2, \mathbb{Z}_3$, then $1-u, \alpha-u$ are units in $End(V_D)$ for any $\alpha \in End(V_D)$.

1. Introduction

Let R be an associative ring with unity. Given a function $f : R \to R$ where R is a noncommutative associative ring with identity, f is said to be *unit-additive* if $f(u+v) = f(u) + f(v)$, for all units $u, v \in R$. Moreover, if $f(uv) = f(u)f(v)$ for all units $u, v \in R$, then the ring R is called *unit-homomorphic* [7]. In [7], the authors proved that every unit additive map of a semilocal ring R is additive if and only if either R has no homomorphic image isomorphic to $\mathbb{Z}_2$ or$R/J(R) \cong \mathbb{Z}_2$ where $J(R)$ denotes the Jacobson radical and $\mathbb{Z}_n$ is the ring of integers modulo n. The study of rings satisfying the 2-sum property (i.e. rings such that each of their elements is a sum of two units) was introduced by Wolfson [12] and Zelinsky [13]. They, independently, proved that the endomorphism ring of a vector space V over a division ring D satisfies the 2-sum property, except that $dim(V) = 1$ and $D = \mathbb{F}_2$. A ring R is said to have *unit sum number n*, if for any $r \in R$ there exist units $u_1, \cdots, u_n$ of R such that $r = u_1 + \cdots + u_n$. According to [8], a ring R is said to satisfy the *binary 2-sum property* if for any $a, b \in R$ there exist units u_1, u_2, u_3 of R such that $a = u_1 + u_2$ and $b = u_1 + u_3$. Recall that a semilocal ring R has unit sum number 2 if and only if no factor ring of R is isomorphic to $\mathbb{F}_2$ (see [5]). Recently, the author of [8] provides a similar

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characterization of semilocal rings with the binary 2-sum property: a semilocal ring R satisfies the binary 2-sum property if and only if no factor ring of R is isomorphic to $\mathbb{F}_2$, $\mathbb{F}_3$, or the 2×2 matrix ring $\mathbb{M}_2(\mathbb{F}_2)$. They also obtained in [8, Corollary 19] that if R is an exchange ring with primitive factors Artinian (e.g., a semilocal ring), then R satisfies the binary 2-sum property if R satisfies the Goodearl–Menal property (Two elements $a, b \in R$ are said to satisfy the Goodearl-Menal condition, in case there exists a unit u in R such that $a - u, u^{-1}$ is a unit. A ring R is said to satisfy the *Gooodearl-Menal* if every elements $a, b \in R$ satisfy this property [6]).

Let V be a countably generated right vector space over a division ring D. In 2010, Chen [3] generalized a result of Zelinsky [13]; it is proved that for any endomorphism f of V there exists an automorphism g of V with $f+g$ and $f-g^{-1}$ both automorphisms of V if $D \neq \mathbb{Z}_2, \mathbb{Z}_3$. We also notice that this result is extended to an Artinian right R-module over a semilocal ring R that contains 1/2 and 1/3. In [10, Theorem], Nicholson and Varadarjan proved that every countable linear transformation over a division ring is clean (every element of a ring is a sum of an idempotent and a unit [9]). Let V be a countably generated vector space over a division ring D such that $|D| \neq 2, 3$, and let $End_D(V)$ denote the ring of linear transformations on V. Chen [4] obtained two interesting decompositions in $End_D(V)$: (1) For any $f \in End_D(V)$, there exists an automorphism g on V such that $f - g$ and $f - g^{-1}$ are both automorphisms on V. Thus, $End_D(V)$ satisfies a special case of the Goodearl-Menal condition. (2) For any $f \in End_D(V)$, there exists an automorphism g on V such that $f^2 - g^2$ is an automorphism on V. In [2], Camillo and Simon also applied The Nicholson-Varadarajan theorem on clean linear transformations and they used the tool of Shift operators.

For a countably infinite dimensional right vector space V_D, a linear transformation $f \in End(V_D)$ is called a *shift operator* if there exists a basis $\{v_1, v_2, \cdots, v_n, \cdots\}$ of V such that $f(v_i) = v_{i+1}$ for all i. Note that the matrix representation of the shift operator f over basis $\{v_i\}_i$ is oof the form

$$f = \begin{pmatrix} 0 & 0 & 0 & 0 & \dots \\ 1 & 0 & 0 & 0 & \dots \\ 0 & 1 & 0 & 0 & \dots \\ 0 & 0 & 1 & 0 & \dots \\ 0 & 0 & 0 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \dots \end{pmatrix}.$$

The main purpos of this study is to obtain the following two generalizations using a new tool, namely *idempotent additive* maps taking idempotents instead of units in a unit additive map: (1) Let V be a countably generated right vector space over a field F and $\sigma \in S = End(V_F)$ be a shift operator. Then there exist a unit $u \in S$ and an idempotent $e \in S$ such that $1-u, \sigma - u$ are units in s and $1-e, \sigma - e$ are idempotents in s. (Theorem 2.4); (2) If D is a division ring and $D \ncong \mathbb{Z}_2, \mathbb{Z}_3$, then there exist a unit $u \in End(V_D)$ for which $1-u, \alpha - u \in U(End(V_D))$ for any $\alpha \in End(V_D)$ (Theorem 2.9).

2. Results

We will denote by $U(R)$ the set of all units and by $Id(R)$ a set of all idempotents of a ring R.

Definition 2.1. Let R be a ring. A map $\sigma : R \to R$ is called an (a) *idempotent* (*unit*) *additive map* if σ is additive on idempotents (units) of R, i.e

$$\sigma(a+b) = \sigma(a) + \sigma(b),$$

for all $a, b \in U(R)$ $(a, b \in Id(R)$.

For convenience, we fix a notation: for $a, b \in R$, we write

$a \leftrightsquigarrow b$ (or $a \overset{u}{\leftrightsquigarrow} b$, to emphasize the element u) if $a-u, b-u \in U(R)$ for some $u \in U(R)$,

$a \rightleftharpoons b$ (or $a \overset{e}{\rightleftharpoons} b$ to emphasize the element e) if $a-e, b-e \in Id(R)$ for some $e \in Id(R)$,

$a \longleftrightarrow b$ (or $a \overset{u}{\longleftrightarrow} b$ to emphasize the unit u), if there exists $u \in U(R)$ such that $a-u, b-u^{-1} \in U(R)$ (Goodearl-Menal condition [6]).

We list some properties of notations in the following observations.

Lemma 2.2. *The followings hold for a ring R and elements $a, b \in R$, $u, x, y \in U(R)$.*

(1) *Let σ be a unit-additive map of R. If $-a \leftrightsquigarrow u$, then $\sigma(a+u) = \sigma(a) + \sigma(u)$.*

(2) *If $1 \leftrightsquigarrow c$ for all $c \in R$, then every unit-additive map of R is additive.*

(3) *Let σ be an automorphism or anti-automorphism of R. Then:*

  (a) $a \overset{u}{\leftrightsquigarrow} b$ *iff* $\sigma(a) \overset{\sigma(u)}{\leftrightsquigarrow} \sigma(b)$.

  (b) $a \overset{u}{\leftrightsquigarrow} b$ *iff* $xay \overset{xuy}{\leftrightsquigarrow} xby$.

(4) (a) $1 \overset{u}{\leftrightsquigarrow} a$ *iff* $1 \overset{u^{-1}}{\longleftrightarrow} a$.

  (b) $1 \leftrightsquigarrow x$ *for all* $x \in R$ *iff* $v \leftrightsquigarrow x$ *for all* $x \in R$ *and all* $v \in U(R)$.

  (c) $1 \longleftrightarrow x$ *for all* $x \in R$ *iff* $v \longleftrightarrow x$ *for all* $x \in R$ *and all* $v \in U(R)$.

  (d) $v \leftrightsquigarrow x$ *for all* $x \in R$ *and all* $v \in U(R)$ *iff* $v \longleftrightarrow x$ *for all* $x \in R$ *and all* $v \in U(R)$.

Proof. (1) and (2) See [7, Lemmas 2.3 and 2.4].
(3) and (4) See [8, Lemmas 2.7 and 2.8]. □

Lemma 2.3. *The followings conditions hold for a ring R and $r \in R$.*

(1) *Let σ be an idempotent-additive map of R. If $e \in Id(R)$ with $-r \rightleftharpoons e$, then $\sigma(r+e) = \sigma(r) + \sigma(e)$.*

(2) *If $1 \rightleftharpoons x$ for all $x \in R$, then every idempotent-additive map of R is additive.*

(3) *$r \rightleftharpoons 1$ if and only if there exist $e, f \in Id(R)$ such that $r = e + f$,*

(4) *Let σ be a ring automorphisms of R. Then $r \rightleftharpoons 1$ if and only if $\sigma(r) \rightleftharpoons 1$*

Proof. (1) and (2) The proofs are similar to the proofs of Lemma 2.2 (1) and (2).
(3) If there exists $e \in Id(R)$ such that $r - e, 1 - e \in Id(R)$, then it is enough to put $f := r - e$. The converse follow from the fact that $1 - e \in Id(R)$ for an arbitrary idempotent e.
(4) This is clear since $\sigma(e) \in Id(R)$ for each $e \in Id(R)$. □

Now we are ready to prove our first main theorem.

Theorem 2.4. *Let V be a countably generated right vector space over a field F and $\sigma \in S = End(V_F)$ be a shift operator. Then*

(1) $1 \rightleftharpoons \sigma$,

(2) $1 \leftrightsquigarrow \sigma$.

Proof. (1) Let $E_1 = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}$, $E_2 = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$, $0_{i\times j}$ is a zero matrix of type $i \times j$ and $(u_i)_{i<\omega}$ be a basis of V. Define an infinite block-diagonal matrices

$$B = \begin{pmatrix} E_1 & 0_{2\times 2} & 0_{2\times 2} & 0_{2\times 2} & \dots \\ 0_{2\times 2} & E_1 & 0_{2\times 2} & 0_{2\times 2} & \dots \\ 0_{2\times 2} & 0_{2\times 2} & E_1 & 0_{2\times 2} & \dots \\ 0_{2\times 2} & 0_{2\times 2} & 0_{2\times 2} & E_1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \dots \end{pmatrix} \text{ and } C = \begin{pmatrix} 0_{1\times 1} & 0_{1\times 2} & 0_{1\times 2} & 0_{1\times 2} & 0_{1\times 2} & \dots \\ 0_{2\times 1} & E_2 & 0_{2\times 2} & 0_{2\times 2} & 0_{2\times 2} & \dots \\ 0_{2\times 1} & 0_{2\times 2} & E_2 & 0_{2\times 2} & 0_{2\times 2} & \dots \\ 0_{2\times 1} & 0_{2\times 2} & 0_{2\times 2} & E_2 & 0_{2\times 2} & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \dots \end{pmatrix},$$

and endomorphisms $e, f \in End(V)$ such that B is the matrix of e and C is the matrix of f with respect to the basis $(u_i)_{i<\omega}$, i.e.

$$e(u_{2i-1}) = e(u_{2i}) = u_{2i},$$

$$f(u_{2i-1}) = 0, \; f(u_{2i}) = u_{2i} + u_{2i+1}$$

for each $i \geq 1$. Then

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \dots \\ 1 & 2 & 0 & 0 & 0 & \dots \\ 0 & 1 & 0 & 0 & 0 & \dots \\ 0 & 0 & 1 & 2 & 0 & \dots \\ 0 & 0 & 0 & 1 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \dots & \end{pmatrix}.$$

is the matrix of $e + f$ and it is easy to see that $e, f \in Id(End(V))$ as $E_1^2 = E_1$ and $E_2^2 = E_2$.

Let us denote $g = e + f$ and we will construct a basis $(v_i)_{i<\omega}$ which witnesses that g is a shift operator, i.e. that $g(v_i) = v_{i+1}$.

First, put $v_1 = u_1$ and $v_2 = u_2$. Then $\mathrm{Span}(v_1, v_2) = \mathrm{Span}(u_1, u_2)$, $g(v_1) = v_2$ and $g(v_2) \in \mathrm{Span}(v_1, v_2, u_3) \setminus \mathrm{Span}(v_1, v_2)$. Let we have constructed $v_1, \dots, v_i$ such that $\mathrm{Span}(v_1, \dots, v_i) = \mathrm{Span}(u_1, \dots, u_i)$, $g(v_{i-1}) = v_i$ and $g(v_i) \in \mathrm{Span}(v_1, \dots, v_i, u_{i+1}) \setminus \mathrm{Span}(v_1, \dots, v_i)$. Then define $v_{i+1} = g(v_i)$. By the induction hypotheses $v_1, \dots, v_{i+1}$ is linearly independent, hence $\mathrm{Span}(v_1, \dots, v_{i+1}) = \mathrm{Span}(u_1, \dots, u_{i+1})$, and it is clear from the matrix A that $g(v_{i+1}) \in \mathrm{Span}(v_1, \dots, v_{i+1}, u_{i+2}) \setminus \mathrm{Span}(v_1, \dots, v_{i+1})$

Since $(v_i)_{i<\omega}$ is a basis satisfying $[e+f](v_i) = v_{i+1}$ for each i, we have proved that $e+f$ is a shift operator, hence $1 \rightleftharpoons e+f$ by Lemma 2.3(3). As there exists an invertible operator, say $a \in End(V)$, such that $e+f = a^{-1}\sigma a$, the assertion follows from Lemma 2.3(4).

(2) Denote by $(v_i)_{i<\omega}$ a basis of V such that $\sigma(v_i) = v_{i+1}$. First, suppose that characteristic of F is not 2. Let $U_1 := \begin{pmatrix} -1 & 0 \\ 1 & -1 \end{pmatrix}$, $U_2 := \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ and $U_3 := \begin{pmatrix} 2 & 0 \\ -1 & 2 \end{pmatrix}$. Remark that all these matrices are invertible. We denote by u an operator such that its matrix with respect to the basis $(v_i)_{i<\omega}$ is

$$[u]_{(v_i)} \begin{pmatrix} U_1 & 0 & 0 & 0 & \dots \\ 0 & U_1 & 0 & 0 & \dots \\ 0 & 0 & U_1 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \dots \end{pmatrix}.$$

Now we easily compute matrices

$$[1-u]_{(v_i)} = \begin{pmatrix} U_3 & 0 & 0 & 0 & \dots \\ 0 & U_3 & 0 & 0 & \dots \\ 0 & 0 & U_3 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \dots \end{pmatrix} \quad \text{and} \quad [\sigma-u]_{(v_i)} = \begin{pmatrix} 1_{1\times 1} & 0 & 0 & 0 & \dots \\ 0 & U_2 & 0 & 0 & \dots \\ 0 & 0 & U_2 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \dots \end{pmatrix}.$$

Since all these matrices are invertible, we can see that $u, 1-u, \sigma-u \in U(S)$.

Now, let $1+1=0$ and consider the matrix

$$A = \begin{pmatrix} U & 0 & 0 & 0 & \dots \\ 0 & U & 0 & 0 & \dots \\ 0 & 0 & U & 0 & \dots \\ 0 & 0 & 0 & U & \dots \\ \vdots & \vdots & \vdots & \vdots & \dots \end{pmatrix},$$

where $U = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix}$ is an invertible matrix with the inverse $U^{-1} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix}$. Clearly, the matrices A and $A+I$ are invertible with the inverses

$$A^{-1} = \begin{pmatrix} U^{-1} & 0 & 0 & 0 & \dots \\ 0 & U^{-1} & 0 & 0 & \dots \\ 0 & 0 & U^{-1} & 0 & \dots \\ 0 & 0 & 0 & U^{-1} & \dots \\ \vdots & \vdots & \vdots & \vdots & \dots \end{pmatrix}$$

and

$$(A+I)^{-1} = \begin{pmatrix} (U+I_3)^{-1} & 0 & 0 & 0 & \dots \\ 0 & (U+I_3)^{-1} & 0 & 0 & \dots \\ 0 & 0 & (U+I_3)^{-1} & 0 & \dots \\ 0 & 0 & 0 & (U+I_3)^{-1} & \dots \\ \vdots & \vdots & \vdots & \vdots & \dots \end{pmatrix},$$

where $(U+I_3)^{-1} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix}$. Let A be the matrix of an operator u with respect to the basis $(v_i)_{i<\omega}$. We have proved that u and $1+u$ are invertible operators.

Finally, the operator $u+\sigma$ is invertible since it has a matrix with respect to $(v_i)_{i<\omega}$

$$\begin{pmatrix} B & 0 & 0 & 0 & \dots \\ E_{13} & B & 0 & 0 & \dots \\ 0 & E_{13} & B & 0 & \dots \\ 0 & 0 & E_{13} & B & \dots \\ \vdots & \vdots & \vdots & \vdots & \dots \end{pmatrix},$$

with the inverse

$$\begin{pmatrix} B^{-1} & 0 & 0 & 0 & \dots \\ C & B^{-1} & 0 & 0 & \dots \\ 0 & C & B^{-1} & 0 & \dots \\ 0 & 0 & C & B^{-1} & \dots \\ \vdots & \vdots & \vdots & \vdots & \dots \end{pmatrix}$$

where $B = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}$, $B^{-1} = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $C = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ and $E_{13} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. □

$GL_n(D)$ denotes the n-dimensional general linear group over a division ring D and $\mathbb{M}_n(D)$ denotes the ring of all $n \times n$ matrices over D with an identity I_n.

Recall that the matrices a and b are *equivalent* if there exists a regular matrix p such that $a = p^{-1}bp$.

Lemma 2.5. *Let D be a division ring of characteristic different from 2, $n \in \mathbb{N}$ and $b \in \mathbb{M}_n(D)$. Then the following conditions are equivalent.*

(1) $b \rightleftharpoons I_n$

(2) *b is equivalent to a block matrix* $\begin{pmatrix} 2I_r & a_{12} & a_{13} & 0 \\ 0 & I_s & a_{23} & 0 \\ 0 & a_{32} & I_t & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \in \mathbb{M}_n(D)$ *where I_r, I_s, I_t are identity matrices, and $a_{i,j}$ and 0 are matrices.*

Proof. Recall that $b \rightleftharpoons I_n$ if and only if there exist $e, f \in Id(\mathbb{M}_n(D))$ such that $b = e + f$ by Lemma 2.3(3). Since

$$\begin{pmatrix} 2I_r & a_{12} & a_{13} & 0 \\ 0 & I_s & a_{23} & 0 \\ 0 & a_{32} & I_t & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} I_r & a_{12} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & a_{32} & I_t & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} I_r & 0 & a_{13} & 0 \\ 0 & I_s & a_{23} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

where both the matrices on the right side are idempotents, we get that (2) $\Rightarrow$ (1) holds.

Let $b = e + f$ for idempotent matrices e, f and let us identify all matrices with linear operator on D^n given by the matrix multiplication. Let us denote by B the basis of $\mathrm{im}(e) \cap \mathrm{im}(f)$ which could be completed to bases of $\mathrm{im}(e)$ and $\mathrm{im}(f)$ by E and F, i.e. $B \cup E$ is a basis of $\mathrm{im}(e)$ and $B \cup F$ is a basis of $\mathrm{im}(f)$. Since e and f are idempotents, we get $e(u) = u$ for each $u \in B \cup E$ and $f(u) = u$ for each $u \in B \cup E$. Hence $e(v) \in \mathrm{Span}(B \cup E)$ and $f(v) \in \mathrm{Span}(B \cup F)$ for all $v \in D^n$.

Finally let K be a basis of $\ker(b)$ and let $k \in \ker(b)$. Then $0 = b(k) = e(k) + f(k)$ and so $e(k) = f(-k) \in \mathrm{im}(e) \cap \mathrm{im}(f) = \mathrm{Span}(B)$. Hence $k = e(k) = f(-k) = -k$ which implies that $k = 0$ and $\ker(b) \subseteq \ker(e) \cap \ker(f)$. It means that the matrix of operator $b = e + f$ with respect to the basis $B \cup E \cup F \cup K$ is of the form

$$\begin{pmatrix} I_r & a_{12} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & a_{32} & I_t & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} I_r & 0 & a_{13} & 0 \\ 0 & I_s & a_{23} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 2I_r & a_{12} & a_{13} & 0 \\ 0 & I_s & a_{23} & 0 \\ 0 & a_{32} & I_t & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

which is equivalent to the matrix b. □

Theorem 2.6. *Let D be a division ring.*

(1) *Let the characteristic of D be different from 2 and $b \in \mathbb{M}_2(D)$. Then $b \rightleftharpoons I_2$ if and only if b is equivalent to one of the matrices:*

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & c \\ d & 1 \end{pmatrix}, \begin{pmatrix} 2 & c \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

for some $c, d \in D$.

(2) *If $D \not\cong \mathbb{Z}_2, \mathbb{Z}_3$ and $n \in \mathbb{N}$, then*

  (i) *for any $a, b \in \mathbb{M}_n(D)$, there exists $c \in GL_n(D)$ such that $b \overset{c}{\leftrightsquigarrow} a$.*

  (ii) $b \overset{c}{\leftrightsquigarrow} I_n$.

Proof. (1) This follows from Lemma 2.5.

(2) Assuming $D \not\cong \mathbb{Z}_2, \mathbb{Z}_3$ implies that $|D| \geq 4$. Let $x, y \in D$. We have the following three cases.

If $x = 0$, then we choose a nonzero element $u \in D$ such that $u \neq y$. Hence $y - u \neq 0$.

If $y = 0$, then we choose a nonzero element $u \in D$ such that $u \neq x$. Hence $x - u \neq 0$.

If $x \neq$ and $y \neq 0$, then we choose a nonzero element $u \in D$ such that $u \neq x$ and $u \neq y$. As a result we obtain that $x \overset{u}{\leftrightsquigarrow} u$.

Let $a = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in \mathbb{M}_n(D)$ and $b = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \in \mathbb{M}_n(D)$, where $a_{11}, b_{11} \in D$, $a_{12}, b_{12} \in \mathbb{M}_{1\times(n-1)}(D)$, $a_{21}, b_{21} \in \mathbb{M}_{(n-1)\times 1}(D)$ and $a_{22}, b_{22} \in \mathbb{M}_{(n-1)\times(n-1)}(D)$. Note that there exists $0 \neq x \in D$ such that $a_{11} - x = u_1 \neq 0$ and $b_{11} - x = u_2 \neq 0$. Since $a_{22} - a_{21}u_1^{-1}a_{12} \in \mathbb{M}_{(n-1)}(D)$ and $b_{22} - b_{21}u_1^{-1}b_{12} \in \mathbb{M}_{(n-1)}(D)$, we can obtain $y \in GL_{n-1}(D)$ such that $a_{22} - a_{21}u_1^{-1}a_{12} - y = v_1 \in GL_{n-1}(D)$ and $b_{22} - b_{21}u_1^{-1}b_{12} - y \in GL_{n-1}(D)$. They imply that

$$a - diag(x, y) = \begin{pmatrix} u_1 & a_{12} \\ a_{21} & v_1 + a_{21}u_1^{-1}a_{12} \end{pmatrix}$$

and

$$b - diag(x, y) = \begin{pmatrix} u_2 & b_{12} \\ b_{21} & v_2 + b_{21}u_1^{-1}b_{12} \end{pmatrix}.$$

Since

$$\begin{pmatrix} u_1 & a_{12} \\ a_{21} & v_1 + a_{21}u_1^{-1}a_{12} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ a_{21}u_1^{-1} & 1 \end{pmatrix} \begin{pmatrix} u_1 & a_{12} \\ 0 & v_1 \end{pmatrix}$$

and

$$\begin{pmatrix} u_2 & b_{12} \\ b_{21} & v_2 + b_{21}u_1^{-1}b_{12} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ b_{21}u_2^{-1} & 1 \end{pmatrix} \begin{pmatrix} u_2 & b_{12} \\ 0 & v_2 \end{pmatrix},$$

we get $\begin{pmatrix} u_1 & a_{12} \\ a_{21} & v_1 + a_{21}u_1^{-1}a_{12} \end{pmatrix}, \begin{pmatrix} u_2 & b_{12} \\ b_{21} & v_2 + b_{21}u_1^{-1}b_{12} \end{pmatrix} \in GL_n(D)$ as desired. □

For the last main theorem we need the following a series of lemmas.

Lemma 2.7. *Let D be a division ring and $\alpha \in End(V_D)$ such that V_D is spanned by $\{y, \alpha(y), \alpha^2(y), \cdots\}$ for some $y \in V$. If $D \ncong \mathbb{Z}_2, \mathbb{Z}_3$, then*

(1) $1 \leftrightsquigarrow \alpha$.

(2) *If V_D is infinitely generated, then* $1 \rightleftharpoons \alpha$.

Proof. (1) We may assume that $V_D \neq 0$. If $\alpha^n(y) \notin yD + \alpha(y)D + \cdots + \alpha^{n-1}(y)D$ for all $n \geq 1$, then $\{y, \alpha(y), \alpha^2(y), \cdots\}$ is a basis of V_D. Since α is a shift operator with respect to the basis $\{y, \alpha(y), \alpha^2(y), \cdots\}$, we get $1 \leftrightsquigarrow \alpha$ by Theorem 2.4(2). Now suppose that there exists $n \in \mathbb{N}$ such that $\alpha^n(y) \notin yD + \alpha(y)D + \cdots + \alpha^{n-1}(y)D$. If n is minimal with respect to this property, then $\{y, \alpha(y), \alpha^2(y), \cdots\}$ forms a basis for V_D. Hence $End_D(V_D) \cong \mathbb{M}_n(D)$.

By Lemma 2.3(2), we obtain that $1 \leftrightsquigarrow \alpha$.

(2) This follows from Theorem 2.4(1) using the arguments of (1) $\square$

Lemma 2.8. *Let D be a division ring such that $D \ncong \mathbb{Z}_2, \mathbb{Z}_3$, $\alpha \in End(V_D)$ and U be an α-invariant subspace of V_D. Assume that there exists a vector $y \in U \setminus V$ such that $V = U + \sum_{i \geq 0} \alpha^i(y)D$. If the restriction $\alpha|_U$ satisfies $1 \leftrightsquigarrow \alpha|_U$, then $1 \leftrightsquigarrow \alpha$*

Proof. Let $V = M \oplus U$ where M is a subspace which contains y. Define

$$\widetilde{\alpha} : V/U \to V/U$$

$$\overline{v} \to \overline{\alpha(v)}$$

(see [10, Lemma 4]). Clearly,

$$\overline{\alpha^n(y)} = \widetilde{\alpha^n}(\overline{v})$$

and there exists a D-subisomorphism $\theta_0 : V/U \to M$ given by $\theta_0(\overline{v}) = \theta(v)$ by [10, Lemma 4] where θ is an idempotent in $End_D(V)$ satisfying $\theta(V) = M$ and $Ker(\theta) = U$. By [10, Lemma 4], we have the endomorphism ring of M as:

$$\beta := \theta_0 \widetilde{\alpha} \theta_0^{-1} : M \to V/U \to V/U \to M.$$

By the hypothesis, $\{\overline{y}, \overline{\alpha(Y)}, \cdots\}$ spans V/U. Hence $\{\overline{y}, \widetilde{\alpha}(\overline{y}), \cdots\}$ spans V/U since $\overline{\alpha^n(y)} = \widetilde{\alpha^n}(\overline{v})$. Now it is easy to see that $\{\theta_0[\overline{y}], \theta_0[\widetilde{\alpha}(\overline{y})], \cdots\}$ spans M. By Lemma 2.7, we get $\beta \leftrightsquigarrow 1$. Then $\beta - v_1 = a_1$ and $1 - v_1 = b_1$ for some units v_1, a_1, b_1 of $End(M)$. By hypothesis, $1 \leftrightsquigarrow \alpha|_U$, we have $\alpha|_U - v_2 = a_2$ and $1 - v_2 = b_2$ for some units v_2, a_2, b_2 of $End(M)$. Since $V = M \oplus U$, we can define

$$v^*(v) = v^*(m + u) = v_1(m) + [\alpha(m) - \beta(m) + v_2(u)].$$

v^* is an automorphism of V: Since $v^*(m + u) = 0$ implies $v_1(m) = 0$ and $[\alpha(m) - \beta(m)] + v_2(u) = 0$, whence $m = u = 0$, we get v^* is monic. As $u = v_2(u_0) = v^*(0 + u_0)$ for some $u_0 \in U$, we obtain $U \subseteq Im(v^*)$. If $m \in M$, we write $m = v_1(m_1)$ for $m_1 \in M$, then $\alpha(m_1) - \beta(m_1) = -v_2(u_0)$. Then $v^*(m_1 + u_0) = v_1(m_1) + [\alpha(m_1) - \beta(m_1) + v_2(u_0)]$ which implies that $M \subseteq Im(v^*)$. Hence v^* is epic.

$\alpha - v^*$ is an automorphism: Firstly,

$$\begin{aligned}(\alpha - v^*)(m+u) &= \alpha(m+u) - v^*(m+u) \\ &= \alpha(m) + \alpha(u) - v_1(m) - [\alpha(m) - \beta(m) - v_2(u)] \\ &= \alpha_{|_u}(u) - v_2(u) - v_1(m) + \beta(m) \\ &= b_2(u) + b_1(m).\end{aligned}$$

Now, by a similar technic of previous proof, we can obtain that $\alpha - v^*$ is monic and epic.

$1 - v^*$ is an automorphism: Firstly,

$$\begin{aligned}(1 - v^*)(m+u) &= 1(m+u) - v^*(m+u) \\ &= \alpha(m) + \alpha(u) - v_1(m) - [\alpha(m) - \beta(m) - v_2(u)] \\ &= 1(m) + 1(u) - v_1(m) - [\alpha(m) - \beta(m) + v_2(u)] \\ &= 1(m) - v_1(m) + 1(u) - v_2(u) + \beta(m) - \alpha(m) \\ &= b_1(m) + [b_2(u) + \beta(m) - \alpha(m)].\end{aligned}$$

Finally, the same argument as for $\alpha - v^*$ shows that $1 - v^*$ is monic and epic. □

Theorem 2.9. *Let D be a division ring and $D \ncong \mathbb{Z}_2, \mathbb{Z}_3$. Then $1 \leftrightsquigarrow \alpha$ for any $\alpha \in End(V_D)$.*

Proof. Fix $\alpha \in End(V_D)$. Define

$$\chi = \{(U, v) : U_D \subseteq V \text{ is a } \ \alpha-\text{invariant and } \alpha_{|_u} \overset{v}{\leftrightsquigarrow} 1\}.$$

Note that $(0, 0) \in \chi$. Now we define $(U, v) \leq (U', v')$ by $U \subseteq U'$ and $v'_{|_u} = v$ is a partial order of χ. By Zorn's Lemma, there exists a maximal element, say (U, v) in χ.

Assume $U \neq V$. Then, take $y \in V \setminus U$ and let $K := \sum_{i \geq 0} \alpha^i(y)D$, and write $V_0 = U + K$. Clearly, V_0 and K are α-invariant subspaces, and $\alpha \in End(V_0)$ and $\alpha_{|_U} \overset{v}{\leftrightsquigarrow} 1$ because $(U, v) \in \chi$. By Lemma 2.8, we get $\alpha \leftrightsquigarrow 1$ which contradicts the maximality of $(U, v) \in \chi$.

□

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