On the structure of ADS and self-small abelian groups

Jan Žemlička

TRTRT 2023, September 16, Cluj-Napoca

Outline

ADS modules

In the sequel
- R denotes an associative ring with unit,
- M a right R-module,
- a group means an abelain group (i.e. $\mathbb{Z}$-module)

A right module M over R is called *absolute direct summand* (ADS) if $M = S \oplus T'$ for every submodules S, T, T' such that $M = S \oplus T$ and T' is a complement of S.

Example

(1) If every idempotent of R is central (in particular if R is commutative or reduced), then R_R is ADS.
(2) Every cyclic module over commutative ring is ADS.

ADS modules

In the sequel

- R denotes an associative ring with unit,
- M a right R-module,
- a group means an abelain group (i.e. $\mathbb{Z}$-module)

A right module M over R is called *absolute direct summand* (ADS) if $M = S \oplus T'$ for every submodules S, T, T' such that $M = S \oplus T$ and T' is a complement of S.

Example

(1) If every idempotent of R is central (in particular if R is commutative or reduced), then R_R is ADS.
(2) Every cyclic module over commutative ring is ADS.

ADS modules

In the sequel
- R denotes an associative ring with unit,
- M a right R-module,
- a group means an abelain group (i.e. $\mathbb{Z}$-module)

A right module M over R is called *absolute direct summand* (ADS) if $M = S \oplus T'$ for every submodules S, T, T' such that $M = S \oplus T$ and T' is a complement of S.

Example

(1) If every idempotent of R is central (in particular if R is commutative or reduced), then R_R is ADS.

(2) Every cyclic module over commutative ring is ADS.

ADS modules

In the sequel
• R denotes an associative ring with unit,
• M a right R-module,
• a group means an abelain group (i.e. $\mathbb{Z}$-module)

A right module M over R is called *absolute direct summand* (ADS) if $M = S \oplus T'$ for every submodules S, T, T' such that $M = S \oplus T$ and T' is a complement of S.

Example

(1) If every idempotent of R is central (in particular if R is commutative or reduced), then R_R is ADS.
(2) Every cyclic module over commutative ring is ADS.

ADS condition via injectivity

A module A is B-*injective* if every homomorphism $C \to A$ for for every submodule $C \leq B$ can be extendede to a homomorphism $B \leq A$.

Theorem (Alahmadi,Jain Leroy, 2012)

The following is equivalent:

1. *M is ADS,*
2. *A and B are mutually injective modules for every $M = A \oplus B$,*
3. *A is a bR-injective module for every $M = A \oplus B$ and $b \in B$.*

Theorem (Alahmadi,Jain Leroy, 2012)

Let R be an simple ring. If R_R is ADS, then either R_R is indecomposable or R is a right self-injective regular ring.

ADS condition via injectivity

A module A is *B-injective* if every homomorphism $C \to A$ for for every submodule $C \leq B$ can be extendede to a homomorphism $B \leq A$.

Theorem (Alahmadi,Jain Leroy, 2012)

The following is equivalent:

1. *M is ADS,*
2. *A and B are mutually injective modules for every $M = A \oplus B$,*
3. *A is a bR-injective module for every $M = A \oplus B$ and $b \in B$.*

Theorem (Alahmadi,Jain Leroy, 2012)

Let R be an simple ring. If R_R is ADS, then either R_R is indecomposable or R is a right self-injective regular ring.

ADS condition via injectivity

A module A is *B-injective* if every homomorphism $C \to A$ for for every submodule $C \leq B$ can be extendede to a homomorphism $B \leq A$.

Theorem (Alahmadi,Jain Leroy, 2012)

The following is equivalent:

1. *M is ADS,*
2. *A and B are mutually injective modules for every $M = A \oplus B$,*
3. *A is a bR-injective module for every $M = A \oplus B$ and $b \in B$.*

Theorem (Alahmadi,Jain Leroy, 2012)

Let R be an simple ring. If R_R is ADS, then either R_R is indecomposable or R is a right self-injective regular ring.

ADS condition via injectivity

A module A is B-*injective* if every homomorphism $C \to A$ for for every submodule $C \leq B$ can be extendede to a homomorphism $B \leq A$.

Theorem (Alahmadi,Jain Leroy, 2012)

The following is equivalent:

1. *M is ADS,*
2. *A and B are mutually injective modules for every $M = A \oplus B$,*
3. *A is a bR-injective module for every $M = A \oplus B$ and $b \in B$.*

Theorem (Alahmadi,Jain Leroy, 2012)

Let R be an simple ring. If R_R is ADS, then either R_R is indecomposable or R is a right self-injective regular ring.

ADS condition via injectivity

A module A is *B-injective* if every homomorphism $C \to A$ for for every submodule $C \leq B$ can be extendede to a homomorphism $B \leq A$.

Theorem (Alahmadi,Jain Leroy, 2012)

The following is equivalent:

1. *M is ADS,*
2. *A and B are mutually injective modules for every $M = A \oplus B$,*
3. *A is a bR-injective module for every $M = A \oplus B$ and $b \in B$.*

Theorem (Alahmadi,Jain Leroy, 2012)

Let R be an simple ring. If R_R is ADS, then either R_R is indecomposable or R is a right self-injective regular ring.

Commuting with $\mathrm{Hom}(M, -)$

Let M be a module and $\mathcal{N}$ a family of abelian groups.
Consider the mapping of abelian groups:

$$\Psi_{\mathcal{N}} : \bigoplus_{N \in \mathcal{N}} \mathrm{Hom}(A, N) \to \mathrm{Hom}(A, \bigoplus \mathcal{N})$$

given by the rule $\Psi_{\mathcal{N}}((f_N)_N) = \sum_N f_N$, where $\sum_N f_N \in \mathrm{Hom}(A, \bigoplus \mathcal{N})$ is defined by $a \to \sum_N f_N(a)$ for f_N viewed as a homomorphism into $\bigoplus \mathcal{N}$.

Lemma

The mapping $\Psi_{\mathcal{N}}$ is an injective homomorphism of abelian groups for every family $\mathcal{N}$.

Commuting with $\mathrm{Hom}(M, -)$

Let M be a module and $\mathcal{N}$ a family of abelian groups. Consider the mapping of abelian groups:

$$\Psi_{\mathcal{N}} : \bigoplus_{N \in \mathcal{N}} \mathrm{Hom}(A, N) \to \mathrm{Hom}(A, \bigoplus \mathcal{N})$$

given by the rule $\Psi_{\mathcal{N}}((f_N)_N) = \sum_N f_N$, where $\sum_N f_N \in \mathrm{Hom}(A, \bigoplus \mathcal{N})$ is defined by $a \to \sum_N f_N(a)$ for f_N viewed as a homomorphism into $\bigoplus \mathcal{N}$.

Lemma

The mapping $\Psi_{\mathcal{N}}$ is an injective homomorphism of abelian groups for every family $\mathcal{N}$.

Commuting with $\mathrm{Hom}(M, -)$

Let M be a module and $\mathcal{N}$ a family of abelian groups. Consider the mapping of abelian groups:

$$\Psi_{\mathcal{N}} : \bigoplus_{N \in \mathcal{N}} \mathrm{Hom}(A, N) \to \mathrm{Hom}(A, \bigoplus \mathcal{N})$$

given by the rule $\Psi_{\mathcal{N}}((f_N)_N) = \sum_N f_N$, where $\sum_N f_N \in \mathrm{Hom}(A, \bigoplus \mathcal{N})$ is defined by $a \to \sum_N f_N(a)$ for f_N viewed as a homomorphism into $\bigoplus \mathcal{N}$.

Lemma

The mapping $\Psi_{\mathcal{N}}$ is an injective homomorphism of abelian groups for every family $\mathcal{N}$.

General relative smallness

Let $\mathcal{C}$ be a class of modules A and B module. We say that A is

- $\mathcal{C}$*-small* if $\Psi_{\mathcal{N}}$ is an isomorphism $\forall\, \mathcal{N} \subseteq \mathcal{C}$,
- B*-small* if it is a $\{B\}$-small module,
- *small* if it is $\mathcal{N}$-small $\forall$ family $\mathcal{N}$ of,
- *self-small* if it is A-small.

Example

(1) Every finitely generated module is small, so self-small.
(2) Let A and B be two modules such that $\mathrm{Hom}(A, B) = 0$. Then A is B-small.
(3) In particular, if $p, q \in \mathbb{P}$ are different primes, A_p is an abelian p-group and A_q is an abelian q-group, then A_p is A_q-small and $\mathbb{Z}$-small.

General relative smallness

Let $\mathcal{C}$ be a class of modules A and B module. We say that A is

- *$\mathcal{C}$-small* if $\Psi_{\mathcal{N}}$ is an isomorphism $\forall\, \mathcal{N} \subseteq \mathcal{C}$,
- *B-small* if it is a $\{B\}$-small module,
- *small* if it is $\mathcal{N}$-small $\forall$ family $\mathcal{N}$ of,
- *self-small* if it is A-small.

Example

(1) Every finitely generated module is small, so self-small.
(2) Let A and B be two modules such that $\mathrm{Hom}(A, B) = 0$. Then A is B-small.
(3) In particular, if $p, q \in \mathbb{P}$ are different primes, A_p is an abelian p-group and A_q is an abelian q-group, then A_p is A_q-small and $\mathbb{Z}$-small.

General relative smallness

Let $\mathcal{C}$ be a class of modules A and B module. We say that A is

- $\mathcal{C}$*-small* if $\Psi_{\mathcal{N}}$ is an isomorphism $\forall\, \mathcal{N} \subseteq \mathcal{C}$,
- B*-small* if it is a $\{B\}$-small module,
- *small* if it is $\mathcal{N}$-small $\forall$ family $\mathcal{N}$ of,
- *self-small* if it is A-small.

Example

(1) Every finitely generated module is small, so self-small.
(2) Let A and B be two modules such that $\mathrm{Hom}(A, B) = 0$. Then A is B-small.
(3) In particular, if $p, q \in \mathbb{P}$ are different primes, A_p is an abelian p-group and A_q is an abelian q-group, then A_p is A_q-small and $\mathbb{Z}$-small.

General relative smallness

Let $\mathcal{C}$ be a class of modules A and B module. We say that A is

- $\mathcal{C}$*-small* if $\Psi_{\mathcal{N}}$ is an isomorphism $\forall\, \mathcal{N} \subseteq \mathcal{C}$,
- B*-small* if it is a $\{B\}$-small module,
- *small* if it is $\mathcal{N}$-small $\forall$ family $\mathcal{N}$ of,
- *self-small* if it is A-small.

Example

(1) Every finitely generated module is small, so self-small.
(2) Let A and B be two modules such that $\mathrm{Hom}(A, B) = 0$. Then A is B-small.
(3) In particular, if $p, q \in \mathbb{P}$ are different primes, A_p is an abelian p-group and A_q is an abelian q-group, then A_p is A_q-small and $\mathbb{Z}$-small.

General relative smallness

Let $\mathcal{C}$ be a class of modules A and B module. We say that A is

- $\mathcal{C}$*-small* if $\Psi_{\mathcal{N}}$ is an isomorphism $\forall\, \mathcal{N} \subseteq \mathcal{C}$,
- B*-small* if it is a $\{B\}$-small module,
- *small* if it is $\mathcal{N}$-small $\forall$ family $\mathcal{N}$ of,
- *self-small* if it is A-small.

Example

(1) Every finitely generated module is small, so self-small.
(2) Let A and B be two modules such that $\mathrm{Hom}(A, B) = 0$. Then A is B-small.
(3) In particular, if $p, q \in \mathbb{P}$ are different primes, A_p is an abelian p-group and A_q is an abelian q-group, then A_p is A_q-small and $\mathbb{Z}$-small.

General relative smallness

Let $\mathcal{C}$ be a class of modules A and B module. We say that A is

- $\mathcal{C}$*-small* if $\Psi_{\mathcal{N}}$ is an isomorphism $\forall\, \mathcal{N} \subseteq \mathcal{C}$,
- B*-small* if it is a $\{B\}$-small module,
- *small* if it is $\mathcal{N}$-small $\forall$ family $\mathcal{N}$ of,
- *self-small* if it is A-small.

Example

(1) Every finitely generated module is small, so self-small.

(2) Let A and B be two modules such that $\mathrm{Hom}(A, B) = 0$. Then A is B-small.

(3) In particular, if $p, q \in \mathbb{P}$ are different primes, A_p is an abelian p-group and A_q is an abelian q-group, then A_p is A_q-small and $\mathbb{Z}$-small.

General relative smallness

Let $\mathcal{C}$ be a class of modules A and B module. We say that A is

- *$\mathcal{C}$-small* if $\Psi_{\mathcal{N}}$ is an isomorphism $\forall\, \mathcal{N} \subseteq \mathcal{C}$,
- *B-small* if it is a $\{B\}$-small module,
- *small* if it is $\mathcal{N}$-small $\forall$ family $\mathcal{N}$ of,
- *self-small* if it is A-small.

Example

(1) Every finitely generated module is small, so self-small.
(2) Let A and B be two modules such that $\mathrm{Hom}(A, B) = 0$. Then A is B-small.
(3) In particular, if $p, q \in \mathbb{P}$ are different primes, A_p is an abelian p-group and A_q is an abelian q-group, then A_p is A_q-small and $\mathbb{Z}$-small.

General relative smallness

Let $\mathcal{C}$ be a class of modules A and B module. We say that A is

- *$\mathcal{C}$-small* if $\Psi_{\mathcal{N}}$ is an isomorphism $\forall\, \mathcal{N} \subseteq \mathcal{C}$,
- *B-small* if it is a $\{B\}$-small module,
- *small* if it is $\mathcal{N}$-small $\forall$ family $\mathcal{N}$ of,
- *self-small* if it is A-small.

Example

(1) Every finitely generated module is small, so self-small.
(2) Let A and B be two modules such that $\mathrm{Hom}(A, B) = 0$. Then A is B-small.
(3) In particular, if $p, q \in \mathbb{P}$ are different primes, A_p is an abelian p-group and A_q is an abelian q-group, then A_p is A_q-small and $\mathbb{Z}$-small.

Criteria of self-smallness

Theorem (Arnold, Murley, 1975)

The following are equivalent for a module A:

1. *A is not self-small,*
2. *$\exists$ an ω-filtration $(A_i \mid i < \omega)$ of A such that $\mathrm{Hom}(A/A_n, A) \neq 0$ for each $n < \omega$,*
3. *$\exists$ an ω-filtration $(A_i \mid i < \omega)$ of A such that for each $n < \omega$ $\exists$ a nonzero $\varphi_n \in \mathrm{End}(A)$ satisfying $\varphi_n(A_n) = 0$.*

Theorem (Albrecht, Breaz, Wickless, 2009)

Let A be a reduced group of finite torsion free rank with a full free subgroup F. The following are equivalent:

1. *A is self-small,*
2. *$\forall$ A_p is finite and $(A/F)_p$ is divisible whenever $A_p \neq 0$,*
3. *each A_p is finite and $\mathrm{Hom}(A, t(A))$ is torsion.*

Criteria of self-smallness

Theorem (Arnold, Murley, 1975)

The following are equivalent for a module A:

1. *A is not self-small,*
2. *$\exists$ an ω-filtration $(A_i \mid i < \omega)$ of A such that $\mathrm{Hom}(A/A_n, A) \neq 0$ for each $n < \omega$,*
3. *$\exists$ an ω-filtration $(A_i \mid i < \omega)$ of A such that for each $n < \omega$ $\exists$ a nonzero $\varphi_n \in \mathrm{End}(A)$ satisfying $\varphi_n(A_n) = 0$.*

Theorem (Albrecht, Breaz, Wickless, 2009)

Let A be a reduced group of finite torsion free rank with a full free subgroup F. The following are equivalent:

1. *A is self-small,*
2. *$\forall$ A_p is finite and $(A/F)_p$ is divisible whenever $A_p \neq 0$,*
3. *each A_p is finite and $\mathrm{Hom}(A, t(A))$ is torsion.*

Criteria of self-smallness

Theorem (Arnold, Murley, 1975)

The following are equivalent for a module A:

1. *A is not self-small,*
2. *$\exists$ an ω-filtration $(A_i \mid i < \omega)$ of A such that $\mathrm{Hom}(A/A_n, A) \neq 0$ for each $n < \omega$,*
3. *$\exists$ an ω-filtration $(A_i \mid i < \omega)$ of A such that for each $n < \omega$ $\exists$ a nonzero $\varphi_n \in \mathrm{End}(A)$ satisfying $\varphi_n(A_n) = 0$.*

Theorem (Albrecht, Breaz, Wickless, 2009)

Let A be a reduced group of finite torsion free rank with a full free subgroup F. The following are equivalent:

1. *A is self-small,*
2. *$\forall$ A_p is finite and $(A/F)_p$ is divisible whenever $A_p \neq 0$,*
3. *each A_p is finite and $\mathrm{Hom}(A, t(A))$ is torsion.*

Criteria of self-smallness

Theorem (Arnold, Murley, 1975)

The following are equivalent for a module A:

1. *A is not self-small,*
2. *$\exists$ an ω-filtration $(A_i \mid i < \omega)$ of A such that $\mathrm{Hom}(A/A_n, A) \neq 0$ for each $n < \omega$,*
3. *$\exists$ an ω-filtration $(A_i \mid i < \omega)$ of A such that for each $n < \omega$ $\exists$ a nonzero $\varphi_n \in \mathrm{End}(A)$ satisfying $\varphi_n(A_n) = 0$.*

Theorem (Albrecht, Breaz, Wickless, 2009)

Let A be a reduced group of finite torsion free rank with a full free subgroup F. The following are equivalent:

1. *A is self-small,*
2. *$\forall$ A_p is finite and $(A/F)_p$ is divisible whenever $A_p \neq 0$,*
3. *each A_p is finite and $\mathrm{Hom}(A, t(A))$ is torsion.*

Criteria of self-smallness

Theorem (Arnold, Murley, 1975)

The following are equivalent for a module A:

1. *A is not self-small,*
2. *$\exists$ an ω-filtration $(A_i \mid i < \omega)$ of A such that $\mathrm{Hom}(A/A_n, A) \neq 0$ for each $n < \omega$,*
3. *$\exists$ an ω-filtration $(A_i \mid i < \omega)$ of A such that for each $n < \omega$ $\exists$ a nonzero $\varphi_n \in \mathrm{End}(A)$ satisfying $\varphi_n(A_n) = 0$.*

Theorem (Albrecht, Breaz, Wickless, 2009)

Let A be a reduced group of finite torsion free rank with a full free subgroup F. The following are equivalent:

1. *A is self-small,*
2. *$\forall$ A_p is finite and $(A/F)_p$ is divisible whenever $A_p \neq 0$,*
3. *each A_p is finite and $\mathrm{Hom}(A, t(A))$ is torsion.*

Criteria of self-smallness

Theorem (Arnold, Murley, 1975)

The following are equivalent for a module A:

1. *A is not self-small,*
2. *$\exists$ an ω-filtration $(A_i \mid i < \omega)$ of A such that $\mathrm{Hom}(A/A_n, A) \neq 0$ for each $n < \omega$,*
3. *$\exists$ an ω-filtration $(A_i \mid i < \omega)$ of A such that for each $n < \omega$ $\exists$ a nonzero $\varphi_n \in \mathrm{End}(A)$ satisfying $\varphi_n(A_n) = 0$.*

Theorem (Albrecht, Breaz, Wickless, 2009)

Let A be a reduced group of finite torsion free rank with a full free subgroup F. The following are equivalent:

1. *A is self-small,*
2. *$\forall$ A_p is finite and $(A/F)_p$ is divisible whenever $A_p \neq 0$,*
3. *each A_p is finite and $\mathrm{Hom}(A, t(A))$ is torsion.*

Examples

Example

If $\mathbb{P}$ denotes the set of all prime numbers, then $M = \prod_{p \in \mathbb{P}} \mathbf{Z}_p$ is a self-small abelian group and $N = \bigoplus_{p \in \mathbb{P}} \mathbf{Z}_p$ is not self-small and $End_R(M) \cong End_R(N)$.

Example

$\prod_{p \in \mathbb{P}} \mathbb{Z}_p$ and $\mathbb{Q}$ are self-small abelian groups. Moreover, $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Q}, \prod_{p \in \mathbb{P}} \mathbb{Z}_p) = \prod_{p \in \mathbb{P}} \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Q}, \mathbb{Z}_p) = 0$. Nevertheless, the product $\mathbb{Q} \times \prod_{p \in \mathbb{P}} \mathbb{Z}_p$ is not self-small.

Examples

Example

If $\mathbb{P}$ denotes the set of all prime numbers, then $M = \prod_{p \in \mathbb{P}} \mathbf{Z}_p$ is a self-small abelian group and $N = \bigoplus_{p \in \mathbb{P}} \mathbf{Z}_p$ is not self-small and $End_R(M) \cong End_R(N)$.

Example

$\prod_{p \in \mathbb{P}} \mathbb{Z}_p$ *and* $\mathbb{Q}$ *are self-small abelian groups. Moreover,* $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Q}, \prod_{p \in \mathbb{P}} \mathbb{Z}_p) = \prod_{p \in \mathbb{P}} \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Q}, \mathbb{Z}_p) = 0$. *Nevertheless, the product* $\mathbb{Q} \times \prod_{p \in \mathbb{P}} \mathbb{Z}_p$ *is not self-small.*

Products of self-small modules

Proposition (Dvořák, 2015)

The following conditions are equivalent for a finite system of self-small modules $(M_i \mid i \leq k)$:

1. $\prod_{i\leq k} M_i$ *is not self-small,*
2. *there exist* $i, j \leq k$ *and a chain* $N_1 \subseteq N_2 \subseteq \ldots \subseteq N_n \subseteq \ldots$ *of proper submodules of* M_i *such that* $\bigcup_{n=1}^{\infty} N_n = M_i$ *and* $\mathrm{Hom}_R(M_i/N_n, M_j) \neq 0$ *for each* $n \in \mathbb{N}$.

Lemma

Let A be an abelian group and $\mathcal{C}$ be a set of abelian groups. Then the following conditions are equivalent:

1. *A is $\bigoplus \mathcal{C}$-small,*
2. *A is $\mathcal{C}$-small,*
3. *A is $\mathrm{Add}(\bigoplus \mathcal{C})$-small.*

Products of self-small modules

Proposition (Dvořák, 2015)

The following conditions are equivalent for a finite system of self-small modules $(M_i \mid i \leq k)$:

1. $\prod_{i \leq k} M_i$ *is not self-small,*
2. *there exist* $i, j \leq k$ *and a chain* $N_1 \subseteq N_2 \subseteq \ldots \subseteq N_n \subseteq \ldots$ *of proper submodules of* M_i *such that* $\bigcup_{n=1}^{\infty} N_n = M_i$ *and* $\mathrm{Hom}_R(M_i/N_n, M_j) \neq 0$ *for each* $n \in \mathbb{N}$*.*

Lemma

Let A be an abelian group and $\mathcal{C}$ be a set of abelian groups. Then the following conditions are equivalent:

1. *A is $\bigoplus \mathcal{C}$-small,*
2. *A is $\mathcal{C}$-small,*
3. *A is $\mathrm{Add}(\bigoplus \mathcal{C})$-small.*

Products of self-small modules

Proposition (Dvořák, 2015)

The following conditions are equivalent for a finite system of self-small modules $(M_i \mid i \leq k)$*:*

1. $\prod_{i \leq k} M_i$ *is not self-small,*
2. *there exist* $i, j \leq k$ *and a chain* $N_1 \subseteq N_2 \subseteq \ldots \subseteq N_n \subseteq \ldots$ *of proper submodules of* M_i *such that* $\bigcup_{n=1}^{\infty} N_n = M_i$ *and* $\mathrm{Hom}_R(M_i/N_n, M_j) \neq 0$ *for each* $n \in \mathbb{N}$*.*

Lemma

Let A *be an abelian group and* $\mathcal{C}$ *be a set of abelian groups. Then the following conditions are equivalent:*

1. A *is* $\bigoplus \mathcal{C}$*-small,*
2. A *is* $\mathcal{C}$*-small,*
3. A *is* $\mathrm{Add}(\bigoplus \mathcal{C})$*-small.*

Products of self-small modules

Proposition (Dvořák, 2015)

The following conditions are equivalent for a finite system of self-small modules $(M_i \mid i \leq k)$:

1. $\prod_{i \leq k} M_i$ *is not self-small,*
2. *there exist* $i, j \leq k$ *and a chain* $N_1 \subseteq N_2 \subseteq \ldots \subseteq N_n \subseteq \ldots$ *of proper submodules of* M_i *such that* $\bigcup_{n=1}^{\infty} N_n = M_i$ *and* $\mathrm{Hom}_R(M_i/N_n, M_j) \neq 0$ *for each* $n \in \mathbb{N}$.

Lemma

Let A be an abelian group and $\mathcal{C}$ be a set of abelian groups. Then the following conditions are equivalent:

1. *A is $\bigoplus \mathcal{C}$-small,*
2. *A is $\mathcal{C}$-small,*
3. *A is $\mathrm{Add}(\bigoplus \mathcal{C})$-small.*

Products of self-small modules

Proposition (Dvořák, 2015)

The following conditions are equivalent for a finite system of self-small modules $(M_i \mid i \leq k)$:

1. $\prod_{i \leq k} M_i$ *is not self-small,*
2. *there exist* $i, j \leq k$ *and a chain* $N_1 \subseteq N_2 \subseteq \ldots \subseteq N_n \subseteq \ldots$ *of proper submodules of* M_i *such that* $\bigcup_{n=1}^{\infty} N_n = M_i$ *and* $\mathrm{Hom}_R(M_i/N_n, M_j) \neq 0$ *for each* $n \in \mathbb{N}$.

Lemma

Let A be an abelian group and $\mathcal{C}$ *be a set of abelian groups. Then the following conditions are equivalent:*

1. *A is* $\bigoplus \mathcal{C}$*-small,*
2. *A is* $\mathcal{C}$*-small,*
3. *A is* $\mathrm{Add}(\bigoplus \mathcal{C})$*-small.*

Torsion-free ADS modules

Lemma

Let M be a torsion-free module over a domain R. Then M is ADS if and only if it is either indecomposable or injective.

Theorem

Let R be a domain. The followings are equivalent for a non-zero torsion-free R-module M and a torsion R-module T:

(1) *$M \oplus T$ is ADS;*

(2) *T is injective and M is either indecomposable or injective.*

Example

An infinite cyclic group is the only example of a finitely generated infinite ADS abelian group.

Torsion-free ADS modules

Lemma

Let M be a torsion-free module over a domain R. Then M is ADS if and only if it is either indecomposable or injective.

Theorem

Let R be a domain. The followings are equivalent for a non-zero torsion-free R-module M and a torsion R-module T:

(1) *$M \oplus T$ is ADS;*

(2) *T is injective and M is either indecomposable or injective.*

Example

An infinite cyclic group is the only example of a finitely generated infinite ADS abelian group.

Torsion-free ADS modules

Lemma

Let M be a torsion-free module over a domain R. Then M is ADS if and only if it is either indecomposable or injective.

Theorem

Let R be a domain. The followings are equivalent for a non-zero torsion-free R-module M and a torsion R-module T:

(1) *$M \oplus T$ is ADS;*

(2) *T is injective and M is either indecomposable or injective.*

Example

An infinite cyclic group is the only example of a finitely generated infinite ADS abelian group.

Torsion-free ADS modules

Lemma

Let M be a torsion-free module over a domain R. Then M is ADS if and only if it is either indecomposable or injective.

Theorem

Let R be a domain. The followings are equivalent for a non-zero torsion-free R-module M and a torsion R-module T:

(1) *$M \oplus T$ is ADS;*

(2) *T is injective and M is either indecomposable or injective.*

Example

An infinite cyclic group is the only example of a finitely generated infinite ADS abelian group.

Mixed ADS groups

Using decomposibility of mixed groups [Kulikov 1941, Th 7] we get:

Lemma

Let A be a non-divisible proper mixed abelian group. Then G is ADS iff $G \cong A \oplus B$ for a non-divisible indecomposable torsion-free and a divisible torsion group B.

Lemma

The following are equivalent for a torsion group G:

(1) *G is ADS,*

(2) $\forall\, p \in \mathbb{P}\ \exists\, n \in \mathbb{N} \cup \{0, \infty\}$ *and* κ *for which* $M_p \cong \mathbb{Z}_{p^n}^{(\kappa)}$.

Example

Let F be a finite group and $n\mathbb{Z} = \mathrm{Ann}(F)$. Then F is ADS if and only if F is a projective $\mathbb{Z}/n\mathbb{Z}$-module.

Mixed ADS groups

Using decomposibility of mixed groups [Kulikov 1941, Th 7] we get:

Lemma

Let A be a non-divisible proper mixed abelian group. Then G is ADS iff $G \cong A \oplus B$ for a non-divisible indecomposable torsion-free and a divisible torsion group B.

Lemma

The following are equivalent for a torsion group G:

(1) *G is ADS,*

(2) $\forall\, p \in \mathbb{P}\ \exists\, n \in \mathbb{N} \cup \{0, \infty\}$ *and* κ *for which* $M_p \cong \mathbb{Z}_{p^n}^{(\kappa)}$.

Example

Let F be a finite group and $n\mathbb{Z} = \mathrm{Ann}(F)$. Then F is ADS if and only if F is a projective $\mathbb{Z}/n\mathbb{Z}$-module.

Mixed ADS groups

Using decomposibility of mixed groups [Kulikov 1941, Th 7] we get:

Lemma

Let A be a non-divisible proper mixed abelian group. Then G is ADS iff $G \cong A \oplus B$ for a non-divisible indecomposable torsion-free and a divisible torsion group B.

Lemma

The following are equivalent for a torsion group G:

(1) *G is ADS,*

(2) $\forall\, p \in \mathbb{P}\ \exists\, n \in \mathbb{N} \cup \{0, \infty\}$ *and* κ *for which* $M_p \cong \mathbb{Z}_{p^n}^{(\kappa)}$.

Example

Let F be a finite group and $n\mathbb{Z} = \mathrm{Ann}(F)$. Then F is ADS if and only if F is a projective $\mathbb{Z}/n\mathbb{Z}$-module.

Mixed ADS groups

Using decomposibility of mixed groups [Kulikov 1941, Th 7] we get:

Lemma

Let A be a non-divisible proper mixed abelian group. Then G is ADS iff $G \cong A \oplus B$ for a non-divisible indecomposable torsion-free and a divisible torsion group B.

Lemma

The following are equivalent for a torsion group G:

(1) *G is ADS,*

(2) $\forall\, p \in \mathbb{P}\ \exists\, n \in \mathbb{N} \cup \{0, \infty\}$ *and* κ *for which* $M_p \cong \mathbb{Z}_{p^n}^{(\kappa)}$.

Example

Let F be a finite group and $n\mathbb{Z} = \mathrm{Ann}(F)$. *Then F is ADS if and only if F is a projective* $\mathbb{Z}/n\mathbb{Z}$*-module.*

Mixed ADS groups

Using decomposibility of mixed groups [Kulikov 1941, Th 7] we get:

Lemma

Let A be a non-divisible proper mixed abelian group. Then G is ADS iff $G \cong A \oplus B$ for a non-divisible indecomposable torsion-free and a divisible torsion group B.

Lemma

The following are equivalent for a torsion group G:

(1) *G is ADS,*

(2) $\forall\, p \in \mathbb{P}\ \exists\ n \in \mathbb{N} \cup \{0, \infty\}$ *and* κ *for which* $M_p \cong \mathbb{Z}_{p^n}^{(\kappa)}$.

Example

Let F be a finite group and $n\mathbb{Z} = \mathrm{Ann}(F)$. Then F is ADS if and only if F is a projective $\mathbb{Z}/n\mathbb{Z}$-module.

Mixed ADS groups

Using decomposibility of mixed groups [Kulikov 1941, Th 7] we get:

Lemma

Let A be a non-divisible proper mixed abelian group. Then G is ADS iff $G \cong A \oplus B$ for a non-divisible indecomposable torsion-free and a divisible torsion group B.

Lemma

The following are equivalent for a torsion group G:

(1) *G is ADS,*

(2) $\forall\, p \in \mathbb{P}\ \exists\, n \in \mathbb{N} \cup \{0, \infty\}$ *and* κ *for which* $M_p \cong \mathbb{Z}_{p^n}^{(\kappa)}$.

Example

Let F be a finite group and $n\mathbb{Z} = \mathrm{Ann}(F)$. Then F is ADS if and only if F is a projective $\mathbb{Z}/n\mathbb{Z}$-module.

Classification of ADS groups

Theorem

An abelian group is ADS if and only if

(1) *either it is divisible,*

(2) *or it is a direct sum of an indecomposable torsion-free group and a divisible torsion group,*

(3) *or it is a torsion group such that each p-component is a direct sum of cyclic p-groups of the same length or of $\mathbb{Z}_{p^\infty}$.*

Example

For an arbitrary cardinal κ there exists 2^κ indecomposable torsion-free abelian groups of cardinality κ, all of which are reduced ADS groups.

Classification of ADS groups

Theorem

An abelian group is ADS if and only if

(1) *either it is divisible,*

(2) *or it is a direct sum of an indecomposable torsion-free group and a divisible torsion group,*

(3) *or it is a torsion group such that each p-component is a direct sum of cyclic p-groups of the same length or of $\mathbb{Z}_{p^\infty}$.*

Example

For an arbitrary cardinal κ there exists 2^κ indecomposable torsion-free abelian groups of cardinality κ, all of which are reduced ADS groups.

Classification of ADS groups

Theorem

An abelian group is ADS if and only if

1. *either it is divisible,*
2. *or it is a direct sum of an indecomposable torsion-free group and a divisible torsion group,*
3. *or it is a torsion group such that each p-component is a direct sum of cyclic p-groups of the same length or of* $\mathbb{Z}_{p^\infty}$.

Example

For an arbitrary cardinal κ there exists 2^κ indecomposable torsion-free abelian groups of cardinality κ, all of which are reduced ADS groups.

Classification of ADS groups

Theorem

An abelian group is ADS if and only if

(1) *either it is divisible,*

(2) *or it is a direct sum of an indecomposable torsion-free group and a divisible torsion group,*

(3) *or it is a torsion group such that each p-component is a direct sum of cyclic p-groups of the same length or of $\mathbb{Z}_{p^\infty}$.*

Example

For an arbitrary cardinal κ there exists 2^κ indecomposable torsion-free abelian groups of cardinality κ, all of which are reduced ADS groups.

Sums vs. products

Lemma

Let $\mathcal{M}$ and $\mathcal{N}$ be finite families of groups. The following conditions are equivalent:

1. $\bigoplus \mathcal{M}$ *is* $\bigoplus \mathcal{N}$*-small,*
2. *M is $\bigoplus \mathcal{N}$-small for each $M \in \mathcal{M}$,*
3. *$\bigoplus \mathcal{M}$ is N-small for each $N \in \mathcal{N}$,*
4. *M is N-small for each $M \in \mathcal{M}$ and $N \in \mathcal{N}$.*

Theorem

Let $\mathcal{M}$ be a family of groups and put $M = \prod \mathcal{M}$ and $S = \bigoplus \mathcal{M}$. Then the following conditions are equivalent:

1. *M is self-small,*
2. *M is S-small,*
3. *M is $\bigoplus \mathcal{C}$-small for each countable family $\mathcal{C} \subseteq \mathcal{M}$.*

Sums vs. products

Lemma

Let $\mathcal{M}$ and $\mathcal{N}$ be finite families of groups. The following conditions are equivalent:

1. $\bigoplus \mathcal{M}$ *is* $\bigoplus \mathcal{N}$*-small,*
2. M *is* $\bigoplus \mathcal{N}$*-small for each* $M \in \mathcal{M}$*,*
3. $\bigoplus \mathcal{M}$ *is* N*-small for each* $N \in \mathcal{N}$*,*
4. M *is* N*-small for each* $M \in \mathcal{M}$ *and* $N \in \mathcal{N}$*.*

Theorem

Let $\mathcal{M}$ be a family of groups and put $M = \prod \mathcal{M}$ and $S = \bigoplus \mathcal{M}$. Then the following conditions are equivalent:

1. M *is self-small,*
2. M *is* S*-small,*
3. M *is* $\bigoplus \mathcal{C}$*-small for each countable family* $\mathcal{C} \subseteq \mathcal{M}$*.*

Sums vs. products

Lemma

Let $\mathcal{M}$ and $\mathcal{N}$ be finite families of groups. The following conditions are equivalent:

1. $\bigoplus \mathcal{M}$ *is* $\bigoplus \mathcal{N}$*-small,*
2. *M is* $\bigoplus \mathcal{N}$*-small for each* $M \in \mathcal{M}$*,*
3. $\bigoplus \mathcal{M}$ *is N-small for each* $N \in \mathcal{N}$*,*
4. *M is N-small for each* $M \in \mathcal{M}$ *and* $N \in \mathcal{N}$*.*

Theorem

Let $\mathcal{M}$ be a family of groups and put $M = \prod \mathcal{M}$ and $S = \bigoplus \mathcal{M}$. Then the following conditions are equivalent:

1. *M is self-small,*
2. *M is S-small,*
3. *M is* $\bigoplus \mathcal{C}$*-small for each countable family* $\mathcal{C} \subseteq \mathcal{M}$*.*

Sums vs. products

Lemma

Let $\mathcal{M}$ and $\mathcal{N}$ be finite families of groups. The following conditions are equivalent:

1. $\bigoplus \mathcal{M}$ *is* $\bigoplus \mathcal{N}$*-small,*
2. *M is* $\bigoplus \mathcal{N}$*-small for each* $M \in \mathcal{M}$*,*
3. $\bigoplus \mathcal{M}$ *is N-small for each* $N \in \mathcal{N}$*,*
4. *M is N-small for each* $M \in \mathcal{M}$ *and* $N \in \mathcal{N}$*.*

Theorem

Let $\mathcal{M}$ be a family of groups and put $M = \prod \mathcal{M}$ and $S = \bigoplus \mathcal{M}$. Then the following conditions are equivalent:

1. *M is self-small,*
2. *M is S-small,*
3. *M is* $\bigoplus \mathcal{C}$*-small for each countable family* $\mathcal{C} \subseteq \mathcal{M}$*.*

Sums vs. products

Lemma

Let $\mathcal{M}$ and $\mathcal{N}$ be finite families of groups. The following conditions are equivalent:

1. $\bigoplus \mathcal{M}$ *is* $\bigoplus \mathcal{N}$*-small,*
2. *M is* $\bigoplus \mathcal{N}$*-small for each* $M \in \mathcal{M}$*,*
3. $\bigoplus \mathcal{M}$ *is N-small for each* $N \in \mathcal{N}$*,*
4. *M is N-small for each* $M \in \mathcal{M}$ *and* $N \in \mathcal{N}$*.*

Theorem

Let $\mathcal{M}$ be a family of groups and put $M = \prod \mathcal{M}$ and $S = \bigoplus \mathcal{M}$. Then the following conditions are equivalent:

1. *M is self-small,*
2. *M is S-small,*
3. *M is* $\bigoplus \mathcal{C}$*-small for each countable family* $\mathcal{C} \subseteq \mathcal{M}$*.*

Sums vs. products

Lemma

Let $\mathcal{M}$ and $\mathcal{N}$ be finite families of groups. The following conditions are equivalent:

1. $\bigoplus \mathcal{M}$ *is* $\bigoplus \mathcal{N}$*-small,*
2. *M is* $\bigoplus \mathcal{N}$*-small for each* $M \in \mathcal{M}$*,*
3. $\bigoplus \mathcal{M}$ *is N-small for each* $N \in \mathcal{N}$*,*
4. *M is N-small for each* $M \in \mathcal{M}$ *and* $N \in \mathcal{N}$*.*

Theorem

Let $\mathcal{M}$ be a family of groups and put $M = \prod \mathcal{M}$ and $S = \bigoplus \mathcal{M}$. Then the following conditions are equivalent:

1. *M is self-small,*
2. *M is S-small,*
3. *M is* $\bigoplus \mathcal{C}$*-small for each countable family* $\mathcal{C} \subseteq \mathcal{M}$*.*

Sums vs. products

Lemma

Let $\mathcal{M}$ and $\mathcal{N}$ be finite families of groups. The following conditions are equivalent:

1. $\bigoplus \mathcal{M}$ *is* $\bigoplus \mathcal{N}$*-small,*
2. *M is* $\bigoplus \mathcal{N}$*-small for each* $M \in \mathcal{M}$,
3. $\bigoplus \mathcal{M}$ *is N-small for each* $N \in \mathcal{N}$,
4. *M is N-small for each* $M \in \mathcal{M}$ *and* $N \in \mathcal{N}$.

Theorem

Let $\mathcal{M}$ *be a family of groups and put* $M = \prod \mathcal{M}$ *and* $S = \bigoplus \mathcal{M}$. *Then the following conditions are equivalent:*

1. *M is self-small,*
2. *M is S-small,*
3. *M is* $\bigoplus \mathcal{C}$*-small for each countable family* $\mathcal{C} \subseteq \mathcal{M}$.

Criterion for mixed groups

Theorem

Let $\mathcal{M}$ be a family of nonzero finitely generated abelian groups such that at least one $N \in \mathcal{M}$ has nonzero torsion part and put $M = \prod \mathcal{M}$, $S = \bigoplus \mathcal{M}$ and $Q = S/T_S$. Then the following conditions are equivalent:

1. *M is self-small,*
2. *S is $\mathbb{Z}$-small and $S_{(p)}$-small for all $p \in \mathbb{P}$,*
3. *$S_{(p)}$ is finite for each $p \in \mathbb{P}$ and Q is finitely generated,*
4. *there are only finitely many $A \in \mathcal{M}$ which are infinite and for each $p \in \mathbb{P}$ there are only finitely many $A \in \mathcal{M}$ with $A_{(p)} \neq 0$,*
5. *the family $\{B \in \mathcal{M} \mid \mathrm{Hom}(B, A) \neq 0\}$ is finite for each $A \in \mathcal{M}$,*
6. *$M \cong F \oplus \prod_{p \in \mathbb{P}} M_p$ for a finitely generated free group F and finite abelian p-groups M_p for each $p \in \mathbb{P}$.*

Criterion for mixed groups

Theorem

Let $\mathcal{M}$ be a family of nonzero finitely generated abelian groups such that at least one $N \in \mathcal{M}$ has nonzero torsion part and put $M = \prod \mathcal{M}$, $S = \bigoplus \mathcal{M}$ and $Q = S/T_S$. Then the following conditions are equivalent:

1. *M is self-small,*
2. *S is $\mathbb{Z}$-small and $S_{(p)}$-small for all $p \in \mathbb{P}$,*
3. *$S_{(p)}$ is finite for each $p \in \mathbb{P}$ and Q is finitely generated,*
4. *there are only finitely many $A \in \mathcal{M}$ which are infinite and for each $p \in \mathbb{P}$ there are only finitely many $A \in \mathcal{M}$ with $A_{(p)} \neq 0$,*
5. *the family $\{B \in \mathcal{M} \mid \mathrm{Hom}(B, A) \neq 0\}$ is finite for each $A \in \mathcal{M}$,*
6. *$M \cong F \oplus \prod_{p \in \mathbb{P}} M_p$ for a finitely generated free group F and finite abelian p-groups M_p for each $p \in \mathbb{P}$.*

Criterion for mixed groups

Theorem

Let $\mathcal{M}$ be a family of nonzero finitely generated abelian groups such that at least one $N \in \mathcal{M}$ has nonzero torsion part and put $M = \prod \mathcal{M}$, $S = \bigoplus \mathcal{M}$ and $Q = S/T_S$. Then the following conditions are equivalent:

1. *M is self-small,*
2. *S is $\mathbb{Z}$-small and $S_{(p)}$-small for all $p \in \mathbb{P}$,*
3. *$S_{(p)}$ is finite for each $p \in \mathbb{P}$ and Q is finitely generated,*
4. *there are only finitely many $A \in \mathcal{M}$ which are infinite and for each $p \in \mathbb{P}$ there are only finitely many $A \in \mathcal{M}$ with $A_{(p)} \neq 0$,*
5. *the family $\{B \in \mathcal{M} \mid \mathrm{Hom}(B, A) \neq 0\}$ is finite for each $A \in \mathcal{M}$,*
6. *$M \cong F \oplus \prod_{p \in \mathbb{P}} M_p$ for a finitely generated free group F and finite abelian p-groups M_p for each $p \in \mathbb{P}$.*

Criterion for mixed groups

Theorem

Let $\mathcal{M}$ be a family of nonzero finitely generated abelian groups such that at least one $N \in \mathcal{M}$ has nonzero torsion part and put $M = \prod \mathcal{M}$, $S = \bigoplus \mathcal{M}$ and $Q = S/T_S$. Then the following conditions are equivalent:

1. *M is self-small,*
2. *S is $\mathbb{Z}$-small and $S_{(p)}$-small for all $p \in \mathbb{P}$,*
3. *$S_{(p)}$ is finite for each $p \in \mathbb{P}$ and Q is finitely generated,*
4. *there are only finitely many $A \in \mathcal{M}$ which are infinite and for each $p \in \mathbb{P}$ there are only finitely many $A \in \mathcal{M}$ with $A_{(p)} \neq 0$,*
5. *the family $\{B \in \mathcal{M} \mid \mathrm{Hom}(B, A) \neq 0\}$ is finite for each $A \in \mathcal{M}$,*
6. *$M \cong F \oplus \prod_{p \in \mathbb{P}} M_p$ for a finitely generated free group F and finite abelian p-groups M_p for each $p \in \mathbb{P}$.*

Criterion for mixed groups

Theorem

Let $\mathcal{M}$ be a family of nonzero finitely generated abelian groups such that at least one $N \in \mathcal{M}$ has nonzero torsion part and put $M = \prod \mathcal{M}$, $S = \bigoplus \mathcal{M}$ and $Q = S/T_S$. Then the following conditions are equivalent:

1. *M is self-small,*
2. *S is $\mathbb{Z}$-small and $S_{(p)}$-small for all $p \in \mathbb{P}$,*
3. *$S_{(p)}$ is finite for each $p \in \mathbb{P}$ and Q is finitely generated,*
4. *there are only finitely many $A \in \mathcal{M}$ which are infinite and for each $p \in \mathbb{P}$ there are only finitely many $A \in \mathcal{M}$ with $A_{(p)} \neq 0$,*
5. *the family $\{B \in \mathcal{M} \mid \mathrm{Hom}(B, A) \neq 0\}$ is finite for each $A \in \mathcal{M}$,*
6. *$M \cong F \oplus \prod_{p \in \mathbb{P}} M_p$ for a finitely generated free group F and finite abelian p-groups M_p for each $p \in \mathbb{P}$.*

Criterion for mixed groups

Theorem

Let $\mathcal{M}$ be a family of nonzero finitely generated abelian groups such that at least one $N \in \mathcal{M}$ has nonzero torsion part and put $M = \prod \mathcal{M}$, $S = \bigoplus \mathcal{M}$ and $Q = S/T_S$. Then the following conditions are equivalent:

1. *M is self-small,*
2. *S is $\mathbb{Z}$-small and $S_{(p)}$-small for all $p \in \mathbb{P}$,*
3. *$S_{(p)}$ is finite for each $p \in \mathbb{P}$ and Q is finitely generated,*
4. *there are only finitely many $A \in \mathcal{M}$ which are infinite and for each $p \in \mathbb{P}$ there are only finitely many $A \in \mathcal{M}$ with $A_{(p)} \neq 0$,*
5. *the family $\{B \in \mathcal{M} \mid \mathrm{Hom}(B, A) \neq 0\}$ is finite for each $A \in \mathcal{M}$,*
6. *$M \cong F \oplus \prod_{p \in \mathbb{P}} M_p$ for a finitely generated free group F and finite abelian p-groups M_p for each $p \in \mathbb{P}$.*

Criterion for mixed groups

Theorem

Let $\mathcal{M}$ be a family of nonzero finitely generated abelian groups such that at least one $N \in \mathcal{M}$ has nonzero torsion part and put $M = \prod \mathcal{M}$, $S = \bigoplus \mathcal{M}$ and $Q = S/T_S$. Then the following conditions are equivalent:

1. *M is self-small,*
2. *S is $\mathbb{Z}$-small and $S_{(p)}$-small for all $p \in \mathbb{P}$,*
3. *$S_{(p)}$ is finite for each $p \in \mathbb{P}$ and Q is finitely generated,*
4. *there are only finitely many $A \in \mathcal{M}$ which are infinite and for each $p \in \mathbb{P}$ there are only finitely many $A \in \mathcal{M}$ with $A_{(p)} \neq 0$,*
5. *the family $\{B \in \mathcal{M} \mid \mathrm{Hom}(B, A) \neq 0\}$ is finite for each $A \in \mathcal{M}$,*
6. *$M \cong F \oplus \prod_{p \in \mathbb{P}} M_p$ for a finitely generated free group F and finite abelian p-groups M_p for each $p \in \mathbb{P}$.*

Structure of products of finitely generated groups

By [Fuchs 1973, Thm 94.2], $\mathbb{Z}$ is slender, hence $\mathrm{Hom}(\mathbb{Z}^\omega, \mathbb{Z}) \cong \mathbb{Z}^{(\omega)}$, from which follows:

Lemma

$\mathbb{Z}^\kappa$ is $\mathbb{Z}$-small, and so self-small for each cardinal κ.

Theorem

Let $\mathcal{M}$ be a family of nonzero finitely generated abelian groups and put $M = \prod \mathcal{M}$, $S = \bigoplus \mathcal{M}$ and $Q = S/T_S$. Then the following conditions are equivalent:

1. *M is self-small,*
2. *either $T_S = 0$, or $S_{(p)}$ is finite for each $p \in \mathbb{P}$ and Q is finitely generated,*
3. *either all $A \in \mathcal{M}$ are free, or the family $\{B \in \mathcal{M} \mid \mathrm{Hom}(B, A) \neq 0\}$ is finite for each $A \in \mathcal{M}$,*
4. *either $M \cong \mathbb{Z}^\kappa$ for some cardinal κ, or $M \cong F \oplus \prod_{p \in \mathbb{P}} M_p$ for a finitely generated free group F and finite abelian p-groups M_p for each $p \in \mathbb{P}$.*

Structure of products of finitely generated groups

By [Fuchs 1973, Thm 94.2], $\mathbb{Z}$ is slender,hence
$\mathrm{Hom}(\mathbb{Z}^\omega, \mathbb{Z}) \cong \mathbb{Z}^{(\omega)}$,from which follows:

Lemma

$\mathbb{Z}^\kappa$ is $\mathbb{Z}$-small, and so self-small for each cardinal κ.

Theorem

Let $\mathcal{M}$ be a family of nonzero finitely generated abelian groups and put $M = \prod \mathcal{M}$, $S = \bigoplus \mathcal{M}$ and $Q = S/T_S$. Then the following conditions are equivalent:

1. *M is self-small,*
2. *either $T_S = 0$, or $S_{(p)}$ is finite for each $p \in \mathbb{P}$ and Q is finitely generated,*
3. *either all $A \in \mathcal{M}$ are free, or the family $\{B \in \mathcal{M} \mid \mathrm{Hom}(B, A) \neq 0\}$ is finite for each $A \in \mathcal{M}$,*
4. *either $M \cong \mathbb{Z}^\kappa$ for some cardinal κ, or $M \cong F \oplus \prod_{p\in\mathbb{P}} M_p$ for a finitely generated free group F and finite abelian p-groups M_p for each $p \in \mathbb{P}$.*

Structure of products of finitely generated groups

By [Fuchs 1973, Thm 94.2], $\mathbb{Z}$ is slender,hence $\mathrm{Hom}(\mathbb{Z}^\omega, \mathbb{Z}) \cong \mathbb{Z}^{(\omega)}$,from which follows:

Lemma

$\mathbb{Z}^\kappa$ is $\mathbb{Z}$-small, and so self-small for each cardinal κ.

Theorem

Let $\mathcal{M}$ be a family of nonzero finitely generated abelian groups and put $M = \prod \mathcal{M}$, $S = \bigoplus \mathcal{M}$ and $Q = S/T_S$. Then the following conditions are equivalent:

1. *M is self-small,*
2. *either $T_S = 0$, or $S_{(p)}$ is finite for each $p \in \mathbb{P}$ and Q is finitely generated,*
3. *either all $A \in \mathcal{M}$ are free, or the family $\{B \in \mathcal{M} \mid \mathrm{Hom}(B, A) \neq 0\}$ is finite for each $A \in \mathcal{M}$,*
4. *either $M \cong \mathbb{Z}^\kappa$ for some cardinal κ, or $M \cong F \oplus \prod_{p \in \mathbb{P}} M_p$ for a finitely generated free group F and finite abelian p-groups M_p for each $p \in \mathbb{P}$.*

Structure of products of finitely generated groups

By [Fuchs 1973, Thm 94.2], $\mathbb{Z}$ is slender,hence $\mathrm{Hom}(\mathbb{Z}^\omega, \mathbb{Z}) \cong \mathbb{Z}^{(\omega)}$,from which follows:

Lemma

$\mathbb{Z}^\kappa$ is $\mathbb{Z}$-small, and so self-small for each cardinal κ.

Theorem

Let $\mathcal{M}$ be a family of nonzero finitely generated abelian groups and put $M = \prod \mathcal{M}$, $S = \bigoplus \mathcal{M}$ and $Q = S/T_S$. Then the following conditions are equivalent:

1. *M is self-small,*
2. *either $T_S = 0$, or $S_{(p)}$ is finite for each $p \in \mathbb{P}$ and Q is finitely generated,*
3. *either all $A \in \mathcal{M}$ are free, or the family $\{B \in \mathcal{M} \mid \mathrm{Hom}(B, A) \neq 0\}$ is finite for each $A \in \mathcal{M}$,*
4. *either $M \cong \mathbb{Z}^\kappa$ for some cardinal κ, or $M \cong F \oplus \prod_{p \in \mathbb{P}} M_p$ for a finitely generated free group F and finite abelian p-groups M_p for each $p \in \mathbb{P}$.*

Structure of products of finitely generated groups

By [Fuchs 1973, Thm 94.2], $\mathbb{Z}$ is slender,hence $\mathrm{Hom}(\mathbb{Z}^\omega, \mathbb{Z}) \cong \mathbb{Z}^{(\omega)}$,from which follows:

Lemma

$\mathbb{Z}^\kappa$ is $\mathbb{Z}$-small, and so self-small for each cardinal κ.

Theorem

Let $\mathcal{M}$ be a family of nonzero finitely generated abelian groups and put $M = \prod \mathcal{M}$, $S = \bigoplus \mathcal{M}$ and $Q = S/T_S$. Then the following conditions are equivalent:

1. *M is self-small,*
2. *either $T_S = 0$, or $S_{(p)}$ is finite for each $p \in \mathbb{P}$ and Q is finitely generated,*
3. *either all $A \in \mathcal{M}$ are free, or the family $\{B \in \mathcal{M} \mid \mathrm{Hom}(B, A) \neq 0\}$ is finite for each $A \in \mathcal{M}$,*
4. *either $M \cong \mathbb{Z}^\kappa$ for some cardinal κ, or $M \cong F \oplus \prod_{p\in\mathbb{P}} M_p$ for a finitely generated free group F and finite abelian p-groups M_p for each $p \in \mathbb{P}$.*

Structure of products of finitely generated groups

By [Fuchs 1973, Thm 94.2], $\mathbb{Z}$ is slender,hence $\mathrm{Hom}(\mathbb{Z}^\omega, \mathbb{Z}) \cong \mathbb{Z}^{(\omega)}$,from which follows:

Lemma

$\mathbb{Z}^\kappa$ is $\mathbb{Z}$-small, and so self-small for each cardinal κ.

Theorem

Let $\mathcal{M}$ be a family of nonzero finitely generated abelian groups and put $M = \prod \mathcal{M}$, $S = \bigoplus \mathcal{M}$ and $Q = S/T_S$. Then the following conditions are equivalent:

1. *M is self-small,*
2. *either $T_S = 0$, or $S_{(p)}$ is finite for each $p \in \mathbb{P}$ and Q is finitely generated,*
3. *either all $A \in \mathcal{M}$ are free, or the family $\{B \in \mathcal{M} \mid \mathrm{Hom}(B, A) \neq 0\}$ is finite for each $A \in \mathcal{M}$,*
4. *either $M \cong \mathbb{Z}^\kappa$ for some cardinal κ, or $M \cong F \oplus \prod_{p \in \mathbb{P}} M_p$ for a finitely generated free group F and finite abelian p-groups M_p for each $p \in \mathbb{P}$.*

Structure of products of finitely generated groups

By [Fuchs 1973, Thm 94.2], $\mathbb{Z}$ is slender,hence $\mathrm{Hom}(\mathbb{Z}^\omega, \mathbb{Z}) \cong \mathbb{Z}^{(\omega)}$,from which follows:

Lemma

$\mathbb{Z}^\kappa$ is $\mathbb{Z}$-small, and so self-small for each cardinal κ.

Theorem

Let $\mathcal{M}$ be a family of nonzero finitely generated abelian groups and put $M = \prod \mathcal{M}$, $S = \bigoplus \mathcal{M}$ and $Q = S/T_S$. Then the following conditions are equivalent:

1. *M is self-small,*
2. *either $T_S = 0$, or $S_{(p)}$ is finite for each $p \in \mathbb{P}$ and Q is finitely generated,*
3. *either all $A \in \mathcal{M}$ are free, or the family $\{B \in \mathcal{M} \mid \mathrm{Hom}(B, A) \neq 0\}$ is finite for each $A \in \mathcal{M}$,*
4. *either $M \cong \mathbb{Z}^\kappa$ for some cardinal κ, or $M \cong F \oplus \prod_{p \in \mathbb{P}} M_p$ for a finitely generated free group F and finite abelian p-groups M_p for each $p \in \mathbb{P}$.*

Structure of products of finitely generated groups

By [Fuchs 1973, Thm 94.2], $\mathbb{Z}$ is slender,hence $\mathrm{Hom}(\mathbb{Z}^\omega, \mathbb{Z}) \cong \mathbb{Z}^{(\omega)}$,from which follows:

Lemma

$\mathbb{Z}^\kappa$ is $\mathbb{Z}$-small, and so self-small for each cardinal κ.

Theorem

Let $\mathcal{M}$ be a family of nonzero finitely generated abelian groups and put $M = \prod \mathcal{M}$, $S = \bigoplus \mathcal{M}$ and $Q = S/T_S$. Then the following conditions are equivalent:

1. *M is self-small,*
2. *either $T_S = 0$, or $S_{(p)}$ is finite for each $p \in \mathbb{P}$ and Q is finitely generated,*
3. *either all $A \in \mathcal{M}$ are free, or the family $\{B \in \mathcal{M} \mid \mathrm{Hom}(B, A) \neq 0\}$ is finite for each $A \in \mathcal{M}$,*
4. *either $M \cong \mathbb{Z}^\kappa$ for some cardinal κ, or $M \cong F \oplus \prod_{p \in \mathbb{P}} M_p$ for a finitely generated free group F and finite abelian p-groups M_p for each $p \in \mathbb{P}$.*

References

- Albrecht, U., Breaz, S., Wickless, W., Self-small Abelian groups. *Bull. Aust. Math. Soc.* **80** (2009), No. 2, 205–216.
- Arnold, D.M., Murley, C.E., Abelian groups, A, such that $\mathrm{Hom}\,(A, -)$ preserves direct sums of copies of A. *Pacific Journal of Mathematics*, Vol. **56** (1975), No.1, 7–20.
- Dvořák, J.: On products of self-small abelian groups, *Stud. Univ. Babeş-Bolyai Math.* 60 (2015), no. 1, 13–17.
- Dvořák, J., Žemlička, J.: Autocompact objects of Ab5 categories, Theory and Applications of Categories, Vol. 37 (2021), 979-995.
- Dvořák, J., Žemlička, J.: Self-small products of abelian groups, Commentat. Math. Univ. Carol., 63,2 (2022) 145—157.
- Fuchs, L.: Infinite Abelian groups. Vol. II. Ac. Press, 1973.
- Koşan, M. T., Žemlička, J.: ADS Abelian groups, to appear in J. Algebra and App.
- Kulikov, L.: On the theory of abelian groups of arbitrary power, Mat. Sbornik 9(1941), 165-182.