4. Coordinate rings

In the sequel: $\boxed{\begin{array}{l} K \text{ is a field with } \bar{K} \text{ the algebraic closure} \\ \underline{X} = \{x_1, \dots, x_n\} \end{array}}$

$\boxed{\text{T\&N}}$ Let $U \subseteq \mathbb{A}^n$ $\quad I_U := \{a \in K[\underline{X}] \mid a(\underline{\alpha}) = 0 \ \forall \underline{\alpha} \in U\}$

$\quad \bar{I}_U := \{a \in \bar{K}[\underline{X}] \mid \text{— " —}\}$

$\underline{\alpha} \in \mathbb{A}^n \quad I_{\underline{\alpha}} := I_{\{\underline{\alpha}\}}, \ \bar{I}_{\underline{\alpha}} := \bar{I}_{\{\underline{\alpha}\}}$

<u>Observation</u>: (1) Let I be an ideal of $K[\underline{X}]$ such that $I \cap K[x_i] = (a_i) \neq 0$, and $d_i := \deg a_i$. Then $K[\underline{X}]/I =$
$= \operatorname{Span}_K(\{x_1^{i_1} \cdots x_n^{i_n} \mid i_j \leq d_j \ \forall j\}) \Rightarrow \dim_K(K[\underline{X}]/I) \leq \prod_{j=1}^n (d_j + 1)$

(2) If R is a K-algebra, $\dim_K(R) < \infty$ and R is a domain
$\Rightarrow \forall \alpha \in R - \{0\}: \alpha^{-1} \in K(\alpha) = K[\alpha] \subseteq R \Rightarrow R$ is a field.

Lemma 4.1 Let $\underline{\alpha} = (\alpha_1, \ldots, \alpha_n) \in \mathbb{A}^n$. Then

(1) $I_{\underline{\alpha}}$ is a maximal ideal.

(2) $\underline{\alpha} \in \mathbb{A}^n(K) \iff I_{\underline{\alpha}}$ is maximal & $K + I_{\alpha} = K[\underline{X}]$

(3) If $\underline{\alpha} \in \mathbb{A}^n(K)$, then $I_{\alpha} = (X_1 - \alpha_1, \ldots, X_n - \alpha_n)$

Proof: Let $\Omega : K[\underline{X}] \to \overline{K}$ is the substitution, i.e.

$$\Omega(a) := a(\underline{\alpha}) \quad \forall a$$

$\Rightarrow \Omega(K[\underline{X}]) = K[\alpha_1, \ldots, \alpha_n] \subseteq \overline{K}$ is a domain

& $\alpha_1, \ldots, \alpha_n$ are algebraic over $K \Rightarrow \dim_K(\Omega(K[\underline{X}])) < \infty$

$\underset{\text{Observation (2)}}{\Longrightarrow} \Omega(K[\underline{X}]) \underset{\text{1. Isomorph. thm}}{\cong} K[\underline{X}]/\ker\Omega$ is a field $\Rightarrow$

$\Rightarrow \ker\Omega = I_{\underline{\alpha}}$ is a maximal ideal ($\Rightarrow$ (1))

(2) $K + I_{\alpha} = K[X] \Leftrightarrow \Omega(K) = K[\alpha_1 \dots \alpha_n] \Leftrightarrow$

$\Leftrightarrow [K[\alpha_1 \dots \alpha_n] : K] = 1 \Leftrightarrow \alpha_i \in K \ \forall i$

(3) follows from (2)

<u>Proposition 4.2</u> If P is a prime ideal of $K[X]$ such that $P \cap K[X_i] \neq 0 \ \forall i$, then $\exists \underline{\alpha} \in \mathbb{A}^n$ for which $P = I_{\underline{\alpha}}$.

<u>Proof</u>: By Observation (1) $\dim_K(K[X]/P) < \infty \Rightarrow$

$\Rightarrow K[X]/P$ is a field by Observation (2) $\Rightarrow P$ maximal

Define $\bar{P} := P\bar{K}[X]$ $(= P\bar{K})$ an ideal of $\bar{K}[X]$

?? Assume that $\bar{P} = \bar{K}[X] \Rightarrow 1 \in \bar{P} \Rightarrow \overset{\text{finitely}}{\exists} \alpha_1 \dots \alpha_n \in \bar{K}$ s.t.

$1 \in P\,K(\alpha_1 \dots \alpha_n)$. Since $[K(\alpha_1 \dots \alpha_n) : K] < \infty \overset{\boxed{1.14}}{\Longrightarrow} P = PK[\alpha_1 \dots \alpha_n][X] \cap K[X]$

As $1 \in P\,K(\alpha_1 \ldots \alpha_n)[\underline{x}] \cap K[\underline{x}] \Rightarrow 1 \in P$ a contradiction

$\Rightarrow \overline{P} \subsetneq \overline{K}[\underline{x}]$

By Zorn's lemma $\exists$ a maximal ideal $M \subsetneq \overline{K}[\underline{x}]$: $P \subseteq \overline{P} \subseteq M$

by 4.1. $\exists \underline{\alpha} \in \mathbb{A}^n : M = \overline{I}_{\underline{\alpha}}$

As $P \subseteq \overline{I}_{\underline{\alpha}} \cap K[\underline{x}] = I_{\underline{\alpha}} \subsetneq K[\underline{x}] \xRightarrow{P\text{-maximal}} P = I_{\underline{\alpha}}$.

<u>Proposition 4.3</u> If P is a prime ideal of $K[x,y]$, then either (0) $P = 0$ or (1) $P = (a)$ for $a \in K[x,y]$ irreducible or (2) P is maximal and $\exists \underline{\alpha} \in \mathbb{A}^2$ such that $P = I_{\underline{\alpha}}$.

<u>Proof</u>: by 2.8 & 4.2

<u>Corollary 4.4</u> Let $0 \neq P \subseteq K[x,y]$ be prime.

(1) P is maximal $\Leftrightarrow \exists \underline{\alpha} \in \mathbb{A}^2 : P = I_{\underline{\alpha}} \Leftrightarrow V_P$ is finite

(2) $\exists a \in K[x,y]$ irreducible : $P = (a) \Leftrightarrow V(a)$ is infinite

(3) if $a, b \in K[x,y]$ are irreducible, $b \notin (a)$, then $V_{(a,b)} = V_a \cap V_b$ is finite.

<u>Example 4.5</u> Let $w = y^2 - (x^3+1) \in \mathbb{R}[x,y]$ (WEP)

then $0 \subseteq (w) \subseteq$ prim ideals

$(y, x+1) = I_{(-1,0)}$

$(y, x^2+x+1) = I_{(e^{\frac{2\pi i}{3}}, 0)}$

$(y^2+1, x) = I_{(0,i)}$

maximal ideals

$\boxed{\text{T\&N}}$ Let $C = V_a$ be an affine planar curve

Recall that $I_C = (a)$

$K[C] := K[x,y]/I_C = K[x,y]/(a)$ is the <u>coordinate ring of C</u>

C is <u>irreducible</u> if $K[C]$ is a domain

If w is WEP, then V_w is called a <u>Weierstrass curve</u>

$p(x,y) + I_C \in K[C]$ is a <u>polynomial on C</u>

<u>Observation</u> Let $C = V_a \subseteq \mathbb{A}^2$ be an affine planar curve, then

(1) C is irreducible $\Leftrightarrow$ $I_C = (a)$ is prime $\Leftrightarrow$

$\Leftrightarrow$ a is irreducible

(2) Let $k[C] \to \{C \to \bar{k}\}$ ~~is~~ defined by the rule $p + (a) \to (\alpha \to p(\alpha))$ is a well-defined injective mapping.

$\boxed{\text{T\&N}}$ If $C = V_{(a)}$ is an irreducible curve, then $K(C) := \left\{ \frac{n + (a)}{d + (a)} \,\middle|\, n \in k[x,y],\ d \in k[x,y] \setminus (a) \right\}$ — the fraction field of $k[C]$ is said to be the <u>function field of C</u>.