$\boxed{L|k \text{ is an AFF over } k \text{ of genus } g,\ \tilde{k} \text{ is the field of constants}}$

Lemma 6.13 Let $\mathcal{S} \subsetneq \mathbb{P}_{L|k}$, $P_1,\dots,P_n \in \mathcal{S}$ be pairwise distinct and $a_1,\dots,a_n \in L$. Then $\forall r\in\mathbb{Z}\ \exists s\in L: v_{P_i}(s-a_i) > r\ \forall i=1,\dots,n$, $v_P(s)\ge 0\ \forall P\in \mathcal{S}\setminus\{P_1,\dots,P_n\}$

Proof: Let $Q\in \mathbb{P}_{L|k}\setminus\mathcal{S}$, define $B_m = \sum_P {}_mb_P P \in Div(L|k)$ such that ${}_mb_Q := m$, ${}_mb_{P_i} := -r-1\ \forall i=1,\dots,n$, $b_P := 0\ \forall P\in\mathbb{P}_{L|k}\setminus\{Q,P_1,\dots,P_n\}$

Comment: we will prove using adèles stronger version of 5.18(2) where all but 1 places P could be suppose to have non-negative value (i.e. $s\in\mathcal{O}_P$)

$\Rightarrow \deg B_m = \deg B_0 + m\underbrace{\deg Q}_{>0} \xRightarrow{6.11} \exists \ell: \forall m\ge \ell\ i(B_m)=0$; put $B := B_\ell$

& define $f\in \mathbb{A}_{L|k}: f(P_i) = a_i\ \forall i=1,\dots,n$, $f(P)=0\ \forall P\in\mathbb{P}_{L|k}\setminus\{P_1,\dots,P_n\} \xRightarrow[\text{as } i(B)=0]{6.12(5)}$

$\Rightarrow \exists s\in L, \exists \tilde f \in \mathbb{A}_{L|k}(B): f = \tilde f - s \Rightarrow s+f = \tilde f\in\mathbb{A}_{L|k}(B)$

$\Rightarrow v_{P_i}(s-a_i)\ge r+1\ \forall i=1,\dots,n$ by the definition of $\mathbb{A}_{L|k}(B)$ & $v_P(s)\ge 0\ \forall P\ne P_i$

Theorem 6.14 (Strong approximation theorem) Let $\mathcal{S}\subsetneq\mathbb{P}_{L|k}$, $P_1,\dots,P_n\in\mathcal{S}$ be pairwise distinct, $a_1,\dots,a_n\in L$ and $r_1,\dots,r_n\in\mathbb{Z}$. Then $\exists s\in L: v_{P_i}(s-a_i) = r_i\ \forall i=1,\dots,n$, $v_P(s)\ge 0\ \forall P\in\mathcal{S}\setminus\{P_1,\dots,P_n\}$

Comment: We can suppose that $s\in\mathcal{O}_P$ for almost all P

Proof: we repeat the arguments of the proof of 5.19 replacing 5.18(2) by 6.13: Let $v_i := v_{P_i}$ and $r := \max\{r_1,\dots,r_n\}$

choose $b_i\in L: v_i(b_i) = r_i \xRightarrow[\text{(2× applied)}]{6.13} \exists t, s\in L: v_P(t)\ge 0, v_P(s)\ge 0\ \forall P\in\mathcal{S}\setminus\{P_1,\dots,P_n\}$ & $v_i(t-b_i) > r\ge r_i$, $v_i(s-(t+a_i)) > r\ge r_i\ \forall i=1,\dots,n$

$\Rightarrow$ compute v_i: $s-a_i = \underbrace{(s-(t+a_i))}_{>r_i} + \underbrace{(t-b_i)}_{>r_i} + \underbrace{b_i}_{=r_i} \Rightarrow v_i(s-a_i) = r_i\ \forall i$. $v_P(s)\ge 0\ \forall P\ne P_i$

7. Weil differentials

Recall: If V is k-space and W its subspace then V^* is the dual space and $W^0 = \{\varphi\in V^*\mid \varphi(W)=0\}$

Def Let $A\in Div(L|k)$

$\Omega_{L|k}(A) := (\mathbb{A}_{L|k}(A)+L)^0_k = \{\varphi\in(\mathbb{A}_{L|k})^*_k \mid \varphi(\mathbb{A}_{L|k}(A)+L)=0\}$

$\Omega_{L|k} := \bigcup\{\Omega_{L|k}(B)\mid B\in Div(L|k)\} = \{\varphi\in(\mathbb{A}_{L|k})^* \mid \varphi(L)=0, \exists B\in Div(L|k): \varphi(\mathbb{A}_{L|k}(B))=0\}$

Elements of $\Omega_{L|k}$ are called Weil differentials.

Comments: Weil differentials are k-linear forms on the space of adèles which vanishes on (constant functions) L and on some $\mathbb{A}_{L|k}(B)$.

Observation Let $A,B\in Div(L|k)$, $s\in L^*$, then:

(1) $\dim_k(\Omega_{L|k}(A)) \overset{1.4(2)}{=} \dim(\mathbb{A}_{L|k}/(\mathbb{A}_{L|k}(A)+L)) \overset{6.12(4)}{=} i(A)$,

(2) if $A\le B \xRightarrow{6.12(1)} \mathbb{A}_{L|k}(A)\subseteq\mathbb{A}_{L|k}(B) \xRightarrow{1.4(4)} \Omega_{L|k}(B)\subseteq\Omega_{L|k}(A)$,

(3) $\Omega_{L|k}(A)\cap\Omega_{L|k}(B) \overset{1.4(3)}{=} (\mathbb{A}_{L|k}(A)+\mathbb{A}_{L|k}(B)+L)^0 \overset{6.12(3)}{=} \Omega_{L|k}(\max(A,B))$

$\Omega_{L|k}(A)+\Omega_{L|k}(B) \overset{1.4(3)}{=} ((\mathbb{A}_{L|k}(A)+L)\cap(\mathbb{A}_{L|k}(B)+L))^0 \overset{1.4(4)}{\subseteq} (\mathbb{A}_{L|k}(A)\cap\mathbb{A}_{L|k}(B)+L)^0 = \Omega_{L|k}(\min(A,B))$,

(4) $\Delta\Omega_{L|K}(A) \overset{1.5(3)}{=} (\Delta^{-1}\mathcal{A}_{L|K}(A))^0 \overset{6.12(6)}{=} \Omega_{L|K}(A+(\Delta))$,

(5) Let $\forall \omega\in\Omega_{L|K}$ define $0\cdot\omega=0$, $(\Delta\cdot\omega)(A)=\omega(\Delta A)$ $\forall\Delta\in L^*$, $\forall A\in L$. Then $\Omega_{L|K}$ is an L-space by (3), (4) & 1.5

Comment (3) $\Rightarrow$ $\Omega_{L|K}$ is again (4) $\Rightarrow$ we have L-multiplication $\Omega_{L|K}$ is also a K-space

Lemma 7.1 Let $\omega\in\Omega_{L|K}\setminus\{0\}$ and $K=\tilde{K}$. Then $\exists!\, W\in Div(L|K)$ such that $\omega(\mathcal{A}_{L|K}(W))=0$ and $\forall A\in Div(L|K)$ satisfying $\omega\in\Omega_{L|K}(A) \to A\le W$

Proof: Note that $\omega\in\Omega_{L|K}(A) \Leftrightarrow \omega(\mathcal{A}_{L|K}(A))=0$ $\forall A\in Div(L|K)$

$\exists A\in Div(L|K)$ such that $\omega\in\Omega_{L|K}(A)$ by the definition of $\Omega_{L|K}$

6.11 says that $\exists\gamma : i(A)=0$ if $A\in Div(L|K)$ of degree $\ge\gamma$

$\Omega_{L|K}(A)\neq 0 \overset{Obs(1)}{\Longrightarrow} \dim_K(\Omega_{L|K}(A)) = i(A)>0$ $\Rightarrow \deg(A)<\gamma$ $\Big\} \Rightarrow$ Fix $W\in Div(L|K)$ of the maximal degree such that $\omega(\mathcal{A}_{L|K}(W))=0$

If $B\in Div(L|K)$ satisfying $\omega\in\Omega_{L|K}(B) \Rightarrow \omega\in\Omega_{L|K}(W)\cap\Omega_{L|K}(B) \overset{Obs.3}{=} \Omega_{L|K}(\max(W,B))$.

$\deg\max(W,B)\le\deg W \Rightarrow B\le W \Rightarrow$ $\boxed{\exists \text{ of } W \text{ is proved}}$

the condition $B\le W$ $\forall B : \omega(\mathcal{A}_{L|K}(B))=0 \Rightarrow$ uniqueness of W

$\boxed{\text{T\&N}}$ The divisor W from 7.1 uniquely determined by the Weil differential $\omega\neq 0$ is called a canonical divisor (of ω) and it is denoted (ω)

Lemma 7.2 Let $\omega,\tilde{\omega}\in\Omega_{L|K}\setminus\{0\}$, $K=\tilde{K}$, $A\in Div(L|K)$, $\psi_\omega : L\to\Omega_{L|K}$ is defined by $\psi_\omega(\Delta)=\Delta\cdot\omega$ $\forall\Delta\in L$. Then:

(1) if $\Delta\in L^* \Rightarrow (\Delta\omega)=(\Delta)+(\omega)$ Comment: (Δ) - principal divisor, (ω) - canonical -"-

(2) ψ_ω is L- and K-linear embedding and $\psi_\omega(\mathcal{L}((\omega)-A))\subseteq\Omega_{L|K}(A)$

(3) $\exists B\in Div(L|K)$ such that $\psi_\omega(\mathcal{L}((\omega)-B))\cap\psi_{\tilde{\omega}}(\mathcal{L}((\tilde{\omega})-B))\neq\{0\}$

Comment: 7.2 is a technical part of the proof that $\Omega_{L|K}$ is $\cong_L L$ (i.e. is an 1-dimensional L-space) and that K-spaces $\Omega_{L|K}(A)$ and $\mathcal{L}((\omega)-A)$ are isomorphic

Proof: (1) Note that $A\le(\Delta\omega) \overset{7.1}{\Longleftrightarrow} \Delta\omega\in\Omega_{L|K}(A) \overset{Obs(4)}{\Longleftrightarrow} \omega\in\Omega_{L|K}(A-(\Delta)) \overset{7.1}{\Longleftrightarrow} A-(\Delta)\le(\omega) \Leftrightarrow A\le(\Delta)+(\omega)$

(a) if we put $A:=(\Delta\omega)$ we get $(\Delta\omega)\le(\Delta)+(\omega)$ (b) if we put $A=(\Delta)+(\omega)$ we get $(\Delta)+(\omega)\le(\Delta\omega)$

$\to (\Delta)+(\omega)=(\Delta\omega)$

(2) Observ.(5) $\Rightarrow$ ψ_ω is L-linear, it is non-trivial $\Rightarrow$ it is injective ($\Leftarrow \dim_L L=1$)

Since $\Delta\in\mathcal{L}((\omega)-A) \Leftrightarrow A\le(\Delta)+(\omega) \overset{(1)}{=} (\Delta\omega) \Leftrightarrow \Delta\omega\in\Omega_{L|K}(A)$

we obtain $\psi_\omega(\mathcal{L}((\omega)-A))\subseteq\Omega_{L|K}(A)$

(3) Let $C\in Div(L|K) : C>0$; (1) $\Rightarrow \psi_\omega(\mathcal{L}((\omega)+C)\subseteq\Omega_{L|K}(-C)$; $-C<0 \overset{(D8)}{\Longrightarrow} \ell(-C)=0$

$\overset{Obs(1)}{\Longrightarrow} \dim_K(\Omega_{L|K}(-C)) = i(-C) = g-1-\deg(-C)+\ell(-C) = g-1+\deg C$

6.11 ensures $\exists\tilde{\gamma}$ such that if $\deg C\ge\tilde{\gamma}$ $i((\omega)+C)=0=i((\tilde{\omega})+C)$

$\Rightarrow \begin{cases} \ell((\omega)+C)=\deg((\omega))+\deg C-g+1 \overset{(2)}{=} \dim_K(\mathcal{L}((\omega)+C)) \\ \ell((\tilde{\omega})+C)=\deg((\tilde{\omega}))+\deg C-g+1 \overset{(2)}{=} \dim_K(\mathcal{L}((\tilde{\omega})+C)) \end{cases}$ for such $\tilde{\gamma}$ & $\deg C\ge\tilde{\gamma}$

Comment: we use 6.11 2× for $\deg((\omega)+C)\to\infty$ & $\deg((\tilde{\omega})+C)\to\infty$

$\psi_\omega(\mathcal{L}((\omega)+C))$, $\psi_{\tilde{\omega}}(\mathcal{L}((\tilde{\omega})+C))$ are K-subspaces of $\Omega_{L|K}(-C)$, if we chose C: $\deg C>3(g-1)-\deg(\omega)-\deg(\tilde{\omega})$ & $\deg C\ge\tilde{\gamma}$

$\Rightarrow \ell((\omega)+C)+\ell((\tilde{\omega})+C)>\dim_K(\Omega_{L|K}(-C)) \overset{L.A}{\Longrightarrow} \psi_\omega(\mathcal{L}((\omega)+C))\cap\psi_{\tilde{\omega}}(\mathcal{L}((\tilde{\omega})+C))\neq\{0\}$

$\boxed{\text{We put } B:=-C}$ $\square$