6. cvičení

1. Srovnejte obory inkluzí a popište všechny jejich prvky
(a) $\mathbb{Z}[\sqrt{6}]$, $\mathbb{Z}[\sqrt{24}]$ a $\mathbb{Z}[\sqrt{2}, \sqrt{3}]$,
(b) $\mathbb{Q}[\sqrt{6}]$, $\mathbb{Q}[\sqrt{24}]$ a $\mathbb{Q}[\sqrt{2}, \sqrt{3}]$,
(c) $\mathbb{Q}[\sqrt[3]{s}]$ a $\mathbb{Q}(\sqrt[3]{s})$ pro libovolné $s \in \mathbb{N}$.

2. Najděte celočíselný polynom stupně 4, jehož kořenem je číslo $\sqrt{2} + \sqrt{3}$ a dokažte, že $\mathbb{Z}[\sqrt{2} + \sqrt{3}] =$

$$= \{p(\sqrt{2}+\sqrt{3}) \mid p \in \mathbb{Z}[x], \deg p \leq 3\} = \{a+b\sqrt{2}+(b+2c)\sqrt{3}+2d\sqrt{6} \mid a, b, c, d \in \mathbb{Z}\}.$$

3. Rozhodněte, zda platí rovnosti
(a) $\mathbb{Z}[\sqrt{2}, \sqrt{3}] = \mathbb{Z}[\sqrt{2} + \sqrt{3}]$, (b) $\mathbb{Q}(\sqrt{2}, \sqrt{3}) = \mathbb{Q}[\sqrt{2} + \sqrt{3}]$.

4. Dokažte, že algoritmus dělení se zbytkem v oboru $\mathbb{Z}[i]$ pracuje správně, tj. pokud pro $u, v \in \mathbb{Z}[i] \setminus 0$ najdeme $a, b \in \mathbb{Q}$, pro která $\frac{u}{v} = a + bi$, a položíme $q = [a] + [b]i \in \mathbb{Z}[i]$ a $z = v - qu$, pak $\nu(z) < \nu(v)$.

Řešení:

1. (a) $\mathbb{Z}[\sqrt{6}] = \{a + b\sqrt{6} |\ a, b \in \mathbb{Z}\}$, $\mathbb{Z}[\sqrt{24}] = \{a + 2b\sqrt{6} |\ a, b \in \mathbb{Z}\}$, $\mathbb{Z}[\sqrt{2}, \sqrt{3}] = \{a + b\sqrt{2} + c\sqrt{3} + d\sqrt{6} |\ a, b, c, d \in \mathbb{Z}\}$,

   $\mathbb{Z}[\sqrt{24}] \subsetneq \mathbb{Z}[\sqrt{6}] \subsetneq \mathbb{Z}[\sqrt{2}, \sqrt{3}]$,

   (b) $\mathbb{Q}[\sqrt{6}] = \mathbb{Q}[\sqrt{24}] = \{a + b\sqrt{6} |\ a, b \in \mathbb{Q}\} \subsetneq \mathbb{Q}[\sqrt{2}, \sqrt{3}] =$

   $= \{a + b\sqrt{2} + c\sqrt{3} + d\sqrt{6} |\ a, b, c, d \in \mathbb{Q}\}$

   (c) $\mathbb{Q}[\sqrt[3]{s}] = \mathbb{Q}(\sqrt[3]{s}) = \{a + b\sqrt[3]{s} + c\sqrt[3]{s^2} |\ a, b, c \in \mathbb{Q}\}$

2. $x^4 - 10x^2 + 1 \Rightarrow \mathbb{Z}[\sqrt{2} + \sqrt{3}] = \{p(\sqrt{2} + \sqrt{3}) |\ p \in \mathbb{Z}[x], \deg p \leq 3\}$,

   $(\sqrt{2} + \sqrt{3})^2 = 5 + 2\sqrt{6}$, $(\sqrt{2} + \sqrt{3})^3 = 11\sqrt{2} + 9\sqrt{3}$

   $\Rightarrow \mathbb{Z}[\sqrt{2} + \sqrt{3}] \subseteq \{a + b\sqrt{2} + (b + 2c)\sqrt{3} + 2d\sqrt{6} |\ a, b, c, d \in \mathbb{Z}\}$,

   $2c\sqrt{3} = 11c(\sqrt{2} + \sqrt{3}) - c(11\sqrt{2} + 9\sqrt{3}) \in \mathbb{Z}[\sqrt{2} + \sqrt{3}], b(\sqrt{2} + \sqrt{3}) \in \mathbb{Z}[\sqrt{2} + \sqrt{3}] \Rightarrow b\sqrt{2} + (b + 2c)\sqrt{3} \in \mathbb{Z}[\sqrt{2} + \sqrt{3}]$,

   $(\sqrt{2} + \sqrt{3})^2 = 5 + 2\sqrt{6} \in \mathbb{Z}[\sqrt{2} + \sqrt{3}] \Rightarrow 2d\sqrt{6} \in \mathbb{Z}[\sqrt{2} + \sqrt{3}]$

   $\Rightarrow \{a + b\sqrt{2} + (b + 2c)\sqrt{3} + 2d\sqrt{6} |\ a, b, c, d \in \mathbb{Z}\} \subseteq \mathbb{Z}[\sqrt{2} + \sqrt{3}]$.

3. (a) ne, (b) ano.

4. $\frac{\|r\|^2}{\|v\|^2} = \|\frac{r}{v}\|^2 = \|\frac{u-qv}{v}\|^2 = \|\frac{u}{v} - q\|^2 = (a - [a])^2 + (b - [b])^2 \leq \frac{1}{4} + \frac{1}{4} \leq \frac{1}{2}$

   $\Rightarrow \nu(r) = \|r\|^2 \leq \frac{1}{2}\|v\|^2 = \nu(v) \Rightarrow \nu(r) < \nu(v)$.