Exercise.

(Easy)

$$\lim_{x\to 0}\frac{x^3-2x}{2x^3+x^2-3x}=\lim_{x\to 0}\frac{x(x^2-2)}{x(2x^2+x-3)}=\lim_{x\to 0}\frac{x^2-2}{2x^2+x-3}\overset{\text{cont.}}{=}\frac{0-2}{0+0-3}=\frac{2}{3}\,.$$

(Medium)

$$\lim_{x\to-\infty}2x(\sqrt{x^2+1}+x)=\lim_{x\to-\infty}2x\left(\underbrace{|x|}_{=-x\text{ for negative }x}\sqrt{1+\frac{1}{x^2}}+x\right)=\lim_{x\to-\infty}2x^2\left(-\sqrt{1+\frac{1}{x^2}}+1\right)$$

$$=\lim_{x\to-\infty}2x^2\left(-\sqrt{1+\frac{1}{x^2}}+1\right)\cdot\frac{\sqrt{1+\frac{1}{x^2}}+1}{\sqrt{1+\frac{1}{x^2}}+1}=\lim_{x\to-\infty}2x^2\cdot\frac{-\left(1+\frac{1}{x^2}\right)+1}{\sqrt{1+\frac{1}{x^2}}+1}$$

$$=\lim_{x\to-\infty}2x^2\cdot\frac{-\frac{1}{x^2}}{\sqrt{1+\frac{1}{x^2}}+1}=\lim_{x\to-\infty}\frac{-2}{\underbrace{\sqrt{1+\frac{1}{x^2}}}_{\to 1,\text{ by (1)}}+1}\overset{\text{AL}}{=}\frac{-2}{1+1}=-1.$$

Ad (1). We used LCC with $g(x)=1+\frac{1}{x^2}$ and $f(y)=\sqrt{y}$. Of course $\lim_{x\to-\infty}g(x)\overset{\text{AL}}{=}1$ and f is continuous at 1, therefore, $\lim_{y\to 1}f(y)=1$.

(Hard)

$$\lim_{x\to 0_+}\sqrt{\frac{1}{x}+\sqrt{\frac{1}{x}+1}}-\sqrt{\frac{1}{x}-\sqrt{\frac{1}{x}+1}}=\lim_{x\to 0_+}\sqrt{\frac{1}{x}+\sqrt{\frac{1}{x}+1}}-\sqrt{\frac{1}{x}-\sqrt{\frac{1}{x}+1}}\cdot\frac{\sqrt{}+\sqrt{}}{\sqrt{}+\sqrt{}}$$

$$=\lim_{x\to 0_+}\frac{\frac{1}{x}+\sqrt{\frac{1}{x}+1}-\left(\frac{1}{x}-\sqrt{\frac{1}{x}+1}\right)}{\sqrt{\frac{1}{x}+\sqrt{\frac{1}{x}+1}}+\sqrt{\frac{1}{x}-\sqrt{\frac{1}{x}+1}}}=\lim_{x\to 0_+}\frac{2\sqrt{\frac{1}{x}+1}}{\sqrt{\frac{1}{x}+\sqrt{\frac{1}{x}+1}}+\sqrt{\frac{1}{x}-\sqrt{\frac{1}{x}+1}}}\,.$$

At this point, we should realize that close to zero (from the right) the most dominant terms are things like $\frac{1}{x}$ (and not 1 or x,...). So,

$$\lim_{x\to 0_+}\frac{2\sqrt{\frac{1}{x}+1}}{\sqrt{\frac{1}{x}+\sqrt{\frac{1}{x}+1}}+\sqrt{\frac{1}{x}-\sqrt{\frac{1}{x}+1}}}=\lim_{x\to 0_+}\frac{2\sqrt{\frac{1}{x}}\cdot\sqrt{1+x}}{\sqrt{\frac{1}{x}}\left[\sqrt{1+x\sqrt{\frac{1}{x}+1}}+\sqrt{1-x\sqrt{\frac{1}{x}+1}}\right]}$$

$$=\lim_{x\to 0_+}\frac{2\sqrt{1+x}}{\sqrt{1+\sqrt{x+x^2}}+\sqrt{1-\sqrt{x+x^2}}}\overset{\text{cont.}}{=}\frac{2\cdot\sqrt{1+0}}{\sqrt{1+\sqrt{0+0}}+\sqrt{1-\sqrt{0+0}}}=\frac{2}{2}=1.$$

<u>Alternative way.</u> You basically use substitution $y=\frac{1}{x}$. We will get much more familiar expressions. But, you need to do that properly, i.e. use LCI. So,

$$\lim_{x\to 0_+}\sqrt{\frac{1}{x}+\sqrt{\frac{1}{x}+1}}-\sqrt{\frac{1}{x}-\sqrt{\frac{1}{x}+1}}\underset{y=\frac{1}{x}}{\overset{(1)}{=}}\lim_{y\to+\infty}\sqrt{y+\sqrt{y+1}}-\sqrt{y-\sqrt{y+1}}$$

$$=\lim_{y\to+\infty}\frac{y+\sqrt{y+1}-(y-\sqrt{y+1})}{\sqrt{y+\sqrt{y+1}}+\sqrt{y-\sqrt{y+1}}}=\lim_{y\to+\infty}\frac{2\sqrt{y}\sqrt{1+\frac{1}{y}}}{\sqrt{y}\left[\sqrt{1+\frac{1}{y}\sqrt{y+1}}+\sqrt{1-\frac{1}{y}\sqrt{y+1}}\right]}$$

$$=\lim_{y\to+\infty}\frac{2\sqrt{1+\frac{1}{y}}}{\sqrt{1+\sqrt{\frac{1}{y}+\frac{1}{y^2}}}+\sqrt{1-\sqrt{\frac{1}{y}+\frac{1}{y^2}}}}\overset{\text{AL}}{=}\frac{2}{1+1}=1.$$

We also need to explain that all square roots go indeed to 1. We need LCC, similarly as in the previous example. Add (1). Let $g(x)=\frac{1}{x}$ and $f(y)=\sqrt{y+\sqrt{y+1}}-\sqrt{y-\sqrt{y+1}}$. Then $\lim_{x\to 0_+}g(x)=+\infty$ (it is important that the limit is just from the right!) and we just computed that $\lim_{y\to+\infty}f(y)=1$. Condition (I) holds trivially, and LCI gives us that $\lim_{x\to 0_+}f(g(x))=1$.