Popular mistakes - Math I

Limit of a sequence.

- You forget to mention using AL. You should do that either in comments or (the most conveniently) above the relevant equality sign. E.g.

$$\lim_{n\to\infty} \frac{1+\frac{1}{n}}{2+\frac{1}{n^3}} \overset{\text{AL}}{=} \frac{1+0}{2+0} = \frac{1}{2}\,.$$

- You write either just

$$\sqrt{1+\frac{1}{n^2}} \to 1$$

or

$$\sqrt{1+\frac{1}{n^2}} \to 1, \text{ by AL.}$$

This is not enough. You need to use RL to get the conclusion, so you have to mention it. And ideally, say that $1+\frac{1}{n^2} \to 1$, by AL and thanks to RL we have also $\sqrt{1+\frac{1}{n^2}} \to 1$.

- You write e.g. that

$$\frac{\sqrt{n}}{n^2} \to 0, \text{ by AL.}$$

This is not optimal, because if you try to use AL at this point, you end up with the undefined expression. First, you should rewrite

$$\frac{\sqrt{n}}{n^2} = \frac{1}{n^{\frac{3}{2}}}$$

and then use AL to get the result.

- You lose "lim" sign during the process. E.g. you write

$$\lim_{n\to\infty} \frac{3+n^3-4n\sqrt{n}}{2n^3-n^2+2} = \frac{n^3}{n^3}\cdot\frac{\frac{3}{n^3}+1-\frac{4n^{3/2}}{n^3}}{2-\frac{n^2}{n^3}+\frac{2}{n^3}} = \frac{\frac{3}{n^3}+1-\frac{4}{n^{3/2}}}{2-\frac{1}{n}+\frac{2}{n^3}} = \frac{0+1-0}{2-0+0} = \frac{1}{2}\,.$$

- Result of a limit depends on the variable. E.g. you have (together with AL and RL) that

$$\lim_{n\to\infty}(\sqrt{n^2+2}-\sqrt{n^2-1}) = \lim_{n\to\infty} \frac{(n^2+2)-(n^2-1)}{\sqrt{n^2}\left(\sqrt{1+\frac{2}{n^2}}+\sqrt{1-\frac{1}{n^2}}\right)}$$
$$= \lim_{n\to\infty}\frac{1}{n}\cdot\frac{3}{\sqrt{1+\frac{2}{n^2}}+\sqrt{1-\frac{1}{n^2}}} = \frac{3}{2n}\,.$$

This is wrong. From the very definition of a limit, the result cannot depend on n. I understand what you are trying to say, but you should write

$$\lim_{n\to\infty}(\sqrt{n^2+2}-\sqrt{n^2-1}) = \cdots = \frac{3}{2}\lim_{n\to\infty}\frac{1}{n}\,.$$

Moreover, this procedure would be, strictly speaking, wrong in that exercise from the homework. You are basically saying that

$$\lim_{n\to\infty} n\cdot(\sqrt{n^2+2}-\sqrt{n^2-1}) = \cdots = (\lim_{n\to\infty} n)\cdot\frac{3}{2}\cdot\left(\lim_{n\to\infty}\frac{1}{n}\right),$$

but this leads to the expression of the type $+\infty\cdot 0$, which is not defined.

To put it correctly you have two options. First, you work with the whole expression (just like in the sample solution). Second, you take the interesting part of the expression and simplifiy it, e.g. you write

$$\sqrt{n^2+2}-\sqrt{n^2-1} = \frac{(n^2+2)-(n^2-1)}{\sqrt{n^2}\left(\sqrt{1+\frac{2}{n^2}}+\sqrt{1-\frac{1}{n^2}}\right)} = \frac{1}{n}\cdot\frac{3}{\sqrt{1+\frac{2}{n^2}}+\sqrt{1-\frac{1}{n^2}}}$$

and then put it together (and comment about AL and RL)

$$\lim_{n\to\infty} n(\sqrt{n^2+2}-\sqrt{n^2-1}) = \lim_{n\to\infty} \frac{3}{\sqrt{1+\frac{2}{n^2}}+\sqrt{1-\frac{1}{n^2}}} = \frac{3}{2}\,.$$

- You work with the expression itself (withou the "lim" sign), but you try to use AL. E.g.

$$\sqrt{n^2+2}-\sqrt{n^2-1} = \frac{(n^2+2)-(n^2-1)}{\sqrt{n^2}\left(\sqrt{1+\frac{2}{n^2}}+\sqrt{1-\frac{1}{n^2}}\right)} = \frac{1}{n}\cdot\frac{3}{\sqrt{1+\frac{2}{n^2}}+\sqrt{1-\frac{1}{n^2}}} = \frac{3}{2n}\,.$$

- Wrong use of AL. In particular, you calculate just a part of a limit and then you try to put it together like this

$$\lim_{n\to\infty} n\cdot(\sqrt{n^2+2}-\sqrt{n^2-1}) = \cdots = (\lim_{n\to\infty} n)\cdot\frac{3}{2}\cdot\left(\lim_{n\to\infty}\frac{1}{n}\right) = \frac{3}{2}\lim_{n\to\infty}\frac{1}{n}\cdot 0 = 0,$$

The same principle is used here

$$1 = \lim_{n\to\infty} 1 = \lim_{n\to\infty} n\cdot\frac{1}{n} = \lim_{n\to\infty} n\cdot 0 = \lim_{n\to\infty} 0 = 0,$$
$$\infty = \lim_{n\to\infty} n = \lim_{n\to\infty} n^2\cdot\frac{1}{n} = \lim_{n\to\infty} n^2\cdot 0 = \lim_{n\to\infty} 0 = 0.$$

You see that these "operations" lead to clearly wrong results. These are the reasons why in the theorem about arithmetics of limits you have the condition "if the RHS is defined". And it also why are certain expressions undefined.

Limit of a function.

- You forget to mention using AL or continuity.
- You do not explain the use of either LCC or LCI.
- Wrong use of AL. Consider e.g. the following calculation:

$$\lim_{x\to 0}\frac{\log(1+x)-x\cos x}{x^2} = \lim_{x\to 0}\frac{\frac{\log(1+x)}{x}-\cos x}{x} \overset{\text{AL}}{=} \lim_{x\to 0}\frac{1-\cos x}{x} = \lim_{x\to 0}\frac{1-\cos x}{x^2}\cdot x \overset{\text{AL}}{=} \frac{1}{2}\cdot 0 = 0.$$

Well, ok, let's try to compute it using L'H:

$$\lim_{x\to 0}\frac{\log(1+x)-x\cos x}{x^2} \overset{\text{L'H},\frac{0}{0}}{=} \lim_{x\to 0}\frac{\frac{1}{1+x}-\cos x+x\sin x}{2x} \overset{\text{L'H},\frac{0}{0}}{=} \lim_{x\to 0}\frac{-\frac{1}{(1+x)^2}+\sin x+\sin x+x\cos x}{2}$$
$$\overset{\text{cont.}}{=} \frac{-1+0+0+0}{2} = -\frac{1}{2}\,.$$

Now, it is obvious that the first computation has to be wrong. The problem is at the second, i.e. the red one, equality. If you want to use AL to simplify the first part of the numerator, you get the following

$$\lim_{x\to 0}\frac{\frac{\log(1+x)}{x}-\cos x}{x} \overset{\text{AL}}{=} \frac{\lim_{x\to 0}\frac{\log(1+x)}{x}-\lim_{x\to 0}\cos x}{\lim_{x\to 0} x} = \frac{1-1}{0} = \frac{0}{0}\,,$$

which is not well-defined; AL is not valid then.

If you check what we did on tutorials (and also sample solutions to homeworks), we used AL during the calculations to simplify e.g. product of several terms; I never did something as above.