VI. Derivatives

Theory.

Definition (Derivative of a function)**.** Let f be a function and $a \in \mathbb{R}$. If the limit

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

exists, then it is called the derivative of the function f at a point a. We denote it by $f'(a)$. We also introduce $f'_+(a)$ and $f'_-(a)$ as a corresponding limit from the right and left, respectively.

Claim (Arithmetics of derivatives)**.** Let the functions f and g have finite derivatives at $a \in \mathbb{R}$. Then

(i) $(\alpha f)'(a) = \alpha f'(a)$, where $\alpha \in \mathbb{R}$.

(ii) $(f+g)'(a) = f'(a) + g'(a)$.

(iii) $(f \cdot g)'(a) = f'(a) \cdot g(a) + f(a) \cdot g'(a)$.

(iv) $\left(\frac{f}{g}\right)'(a) = \frac{f'(a) \cdot g(a) - f(a) \cdot g'(a)}{g^2(a)}$, provided that $g(a) \neq 0$.

Claim (Derivative of a compound function)**.** Suppose that the function f has a finite derivative at $y_0 \in \mathbb{R}$, the function g has a finite derivative at $x_0 \in \mathbb{R}$, and $y_0 = g(x_0)$. Then

$$(f \circ g)'(x_0) = f'(y_0) \cdot g'(x_0).$$

Claim (Computation of one-sided derivative)**.** Suppose that a function f is continuous from the right at $a \in \mathbb{R}$ and the limit $\lim_{x \to a_+} f'(x)$ exists. Then the derivative $f'_+(a)$ exists and there holds

$$f'_+(a) = \lim_{x \to a_+} f'(x).$$

The left-side variant is analogous.

Claim (L'Hôpital's rule)**.** Suppose that functions f and g have finite derivatives on some punctured neighbourhood of $a \in \mathbb{R}^*$ and the limit $\lim_{x \to a} \frac{f'(x)}{g'(x)}$ exists. Suppose further that one of the following conditions hold:

(i) $\lim_{x \to a} f(x) = \lim_{x \to a} g(x) = 0$.

(ii) $\lim_{x \to a} |g(x)| = +\infty$.

Then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}.$$

Claim (Derivatives of elementary functions)**.**

(i) $1' = 0$, $x \in \mathbb{R}$.

(ii) $(x^n)' = nx^{n-1}$, $x \in \mathbb{R}$, $n \in \mathbb{N}$.

(iii) $(x^{-n})' = -nx^{-n-1}$, $x \in \mathbb{R} \setminus \{0\}$, $n \in \mathbb{N}$.

(iv) $(a^x)' = a^x \log a$, $x \in \mathbb{R}$, $a > 0$.

(v) $(\log x)' = \frac{1}{x}$, $x > 0$.

(vi) $(x^\alpha)' = \alpha x^{\alpha-1}$, $x > 0$, $\alpha \in \mathbb{R}$.

(vii) $(\sin x)' = \cos x$, $x \in \mathbb{R}$.

(viii) $(\cos x)' = -\sin x$, $x \in \mathbb{R}$.

(ix) $(\tan x)' = \frac{1}{\cos^2 x}$, $x \in \bigcup_{k \in \mathbb{Z}} \left(-\frac{\pi}{2} + k\pi, \frac{\pi}{2} + k\pi\right)$.

(x) $(\arcsin x)' = \frac{1}{\sqrt{1-x^2}}$, $x \in (-1, 1)$.

(xi) $(\arccos x)' = -\frac{1}{\sqrt{1-x^2}}$, $x \in (-1, 1)$.

(xii) $(\arctan x)' = \frac{1}{1+x^2}$, $x \in \mathbb{R}$.

Exercise 1 (Definition)**.** Find f' using the definition, determine $\mathcal{D}_f$ and $\mathcal{D}_{f'}$.

(a) $f(x) = 8$.
(b) $f(x) = x^3$.
(c) $f(x) = \frac{1}{x^2}$.
(d) $f(x) = \sqrt{x}$.
(e) $f(x) = 3\sqrt[3]{x}$.
(f) $f(x) = |x|$.
(g) $f(x) = e^x$.
(h) $f(x) = 2^x$.
(i) $f(x) = \log x$.
(j) $f(x) = \sin x$.
(k) $f(x) = \tan x$.
(l) $f(x) = \arctan x$.

Exercise 2 (Arithmetics of derivatives)**.** Find f' and determine also $\mathcal{D}_f$ and $\mathcal{D}_{f'}$.

(a) $f(x) = 3x^5 - 17x^3 + \sqrt{3}x - 8$.
(b) $f(x) = (x^2 - 2x + 3)e^x$.
(c) $f(x) = x + 2\sqrt{x} + \sqrt[4]{x}$.
(d) $f(x) = x^{\frac{3}{4}} - \frac{4}{\sqrt{x}}$.
(e) $f(x) = x \log x + \frac{1}{x^2} \log x$.
(f) $f(x) = \left(\frac{3}{2}\right)^x + e^x$.
(g) $f(x) = x^2 \cos x$.
(h) $f(x) = \sin x \cdot \arctan x$.
(i) $f(x) = e^x \tan x + e^2$.
(j) $f(x) = \log x + \frac{3^x}{x^3}$.
(k) $f(x) = \frac{x^3}{e^x}$.
(l) $f(x) = \frac{x-1}{1+x^2}$.
(m) $f(x) = \frac{1}{\tan x}$.
(n) $f(x) = \frac{\sin x}{x^2+3x-4}$.
(o) $f(x) = \frac{2^x}{x^2-1}$.

Exercise 3 (Compound functions)**.** Find f' and determine also $\mathcal{D}_f$ and $\mathcal{D}_{f'}$.

(a) $f(x) = (x^2 + 51x + 119)^{87}$.
(b) $f(x) = x^3(x+2)^8(x-7)^{11}$.
(c) $f(x) = \log(x^2 + x + 2)$.
(d) $f(x) = \sin^2(3x) + \log^3(2x)$.
(e) $f(x) = e^{-\frac{1}{x^2}}$.
(f) $f(x) = \sin^2 x - \sin(x^2)$.
(g) $f(x) = \log(\arctan x)$.
(h) $f(x) = 2 \log \frac{x^2-1}{x^2+1}$.
(i) $f(x) = \arctan \sqrt{x}$.
(j) $f(x) = \cos^3(x^3 - x + 2)^9$.
(k) $f(x) = \frac{x}{\sqrt{9-x^2}}$.
(l) $f(x) = \frac{x^2(1-x)^3}{1+x}$.
(m) $f(x) = \frac{1}{\sqrt{3}} \arctan \frac{\sqrt{3}}{x}$.
(n) $f(x) = x \arcsin \sqrt{x}$.
(o) $f(x) = \sin(\sin(\sin x))$.
(p) $f(x) = \log^2(\log^3 x)$.
(q) $f(x) = x^x$.
(r) $f(x) = \left(\frac{1}{x}\right)^{\frac{1}{x}}$.
(s) $f(x) = (\sin x)^{\cos x}$.
(t) $f(x) = (x+1) \cdot \arccos \frac{1}{x^2+1}$.
(u) $f(x) = e^{x^2-1} - \log \sqrt{\frac{e^{2x}}{e^{2x}+1}}$.

Exercise 4 (One-sided derivatives)**.** Find f' and determine also the domains of f and f'.

(a) $f(x) = \begin{cases} x, & x \in (-\infty, 0] \\ \log(1+x), & x \in (0, \infty) \end{cases}$.
(b) $f(x) = \begin{cases} 1-x, & x \in (-\infty, 1) \\ (1-x)(2-x), & x \in [1, 2] \\ -(2-x), & x \in (2, \infty) \end{cases}$.
(c) $f(x) = \max\{1 - x^2, (x-1)^2\}$.
(d) $f(x) = x|x| + 1$.
(e) $f(x) = e^{(x-2)^2} \operatorname{sign}(x^2 - 4)$.
(f) $f(x) = |\log|x||$.
(g) $f(x) = \arcsin(x^2 - 1)$.
(h) $f(x) = \cos(\max\{x, x^2\})$.

Exercise 5 (L'Hôpital's rule)**.** Find limits.

(a) $\lim_{x\to 0} \frac{\sin 2x}{\tan 4x}$.
(b) $\lim_{x\to +\infty} \frac{\log x^2}{3x+5}$.
(c) $\lim_{x\to +\infty} \frac{e^{3x}}{3-2x}$.
(d) $\lim_{x\to 0} \frac{1-\cos x}{x^2}$.
(e) $\lim_{x\to +\infty} \frac{x^3+3x^2+1}{5x^3+6x^2+3x}$.
(f) $\lim_{x\to +\infty} \frac{e^{7x}}{2x^3-x^2+x+1}$.
(g) $\lim_{x\to 0} \frac{\log \cos x}{x^2}$.
(h) $\lim_{x\to 0} \frac{1+\sin x-\cos x}{1-\sin x-\cos x}$.
(i) $\lim_{x\to 2} \frac{x^2+2x-8}{x^2+x-6}$.
(j) $\lim_{x\to +\infty} \frac{x^2-x+3}{\log^2 x}$.
(k) $\lim_{x\to 4} \frac{x-4}{\sqrt{x-3}-1}$.
(l) $\lim_{x\to 2} \frac{e-e^{x-1}}{2x-4}$.
(m) $\lim_{x\to 0} \frac{\sin 4x-\sin 2x}{\tan 3x}$.
(n) $\lim_{x\to 0} \frac{\cos x-\cos 3x}{x^2}$.
(o) $\lim_{x\to -\infty} x^3 e^{2x}$.
(p) $\lim_{x\to 0} \frac{\arcsin x}{\sin(\arctan x)}$.
(q) $\lim_{x\to 0_+} x \log x$.
(r) $\lim_{x\to +\infty} x \sin \frac{1}{x}$.
(s) $\lim_{x\to 0} \frac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt[3]{1+x}-\sqrt[3]{1-x}}$.
(t) $\lim_{x\to +\infty} \frac{\log(x^3-\arctan x)}{\log(x^2+\arctan x)}$.
(u) $\lim_{x\to +\infty} \frac{\log(1+3^x)}{\log(1+2^x)}$.

Results - VI. Derivatives

Exercise 1 (Definition)**.**

(a) $f'(x) = 0$, $x \in \mathbb{R}$.

(b) $f'(x) = 3x^2$, $x \in \mathbb{R}$.

(c) $f'(x) = -\frac{2}{x^3}$, $x \in (-\infty, 0) \cup (0, +\infty)$.

(d) $f'(x) = \begin{cases} \frac{1}{2\sqrt{x}}, & x > 0 \\ +\infty, & x = 0_+ \end{cases}$.

(e) $f'(x) = \begin{cases} \frac{1}{\sqrt[3]{x^2}}, & x \neq 0 \\ +\infty, & x = 0 \end{cases}$.

(f) $f'(x) = \begin{cases} 1, & x > 0 \\ \text{Does not exist}, & x = 0 \\ -1, & x < 0 \end{cases}$.

(g) $f'(x) = e^x$, $x \in \mathbb{R}$.

(h) $f'(x) = 2^x \log 2$, $x \in \mathbb{R}$.

(i) $f'(x) = \frac{1}{x}$, $x > 0$.

(j) $f'(x) = \cos x$, $x \in \mathbb{R}$.

(k) $f'(x) = \frac{1}{\cos^2 x}$, $x \in \left(-\frac{\pi}{2} + k\pi, \frac{\pi}{2} + k\pi\right)$, $k \in \mathbb{Z}$.

(l) $f'(x) = \frac{1}{1+x^2}$, $x \in \mathbb{R}$.

Exercise 2 (Elementary)**.**

(a) $f'(x) = 15x^4 - 51x^2 + \sqrt{3}$, $x \in \mathbb{R}$.

(b) $f'(x) = (x^2 + 1)e^x$, $x \in \mathbb{R}$.

(c) $f'(x) = 1 + \frac{1}{\sqrt{x}} + \frac{1}{4\sqrt[4]{x^3}}$, $x > 0$.

(d) $f'(x) = \frac{3}{4\sqrt[4]{x}} + \frac{2}{\sqrt{x^3}}$, $x > 0$.

(e) $f'(x) = \left(1 - \frac{2}{x^3}\right) \log x + 1 + \frac{1}{x^3}$, $x > 0$.

(f) $f'(x) = \left(\frac{3}{2}\right)^x \log \frac{3}{2} + e^x$, $x \in \mathbb{R}$.

(g) $f'(x) = 2x \cos x - x^2 \sin x$, $x \in \mathbb{R}$.

(h) $f'(x) = \cos x \cdot \arctan x + \frac{\sin x}{1+x^2}$, $x \in \mathbb{R}$.

(i) $f'(x) = e^x \tan x + \frac{e^x}{\cos^2 x}$, $x \in \left(-\frac{\pi}{2} + k\pi, \frac{\pi}{2} + k\pi\right)$, $k \in \mathbb{Z}$.

(j) $f'(x) = \frac{1}{x} + \frac{3^x \log 3 \cdot x - 3^{x+1}}{x^4}$, $x > 0$.

(k) $f'(x) = \frac{3x^2 - x^3}{e^x}$, $x \in \mathbb{R}$.

(l) $f'(x) = \frac{1 - x^2 + 2x}{(1+x^2)^2}$, $x \in \mathbb{R}$.

(m) $f'(x) = -\frac{1}{\sin^2 x}$, $x \in (k\pi, \pi + k\pi)$, $k \in \mathbb{Z}$.

(n) $f'(x) = \frac{(x^2+3x-4)\cos x - (2x+3)\sin x}{(x^2+3x-4)^2}$, $x \in (-\infty, -4) \cup (-4, 1) \cup (1, +\infty)$.

(o) $f'(x) = \frac{(x^2-1)\log 2 \cdot 2^x - x \cdot 2^{x+1}}{(x^2-1)^2}$, $x \in (-\infty, -1) \cup (-1, 1) \cup (1, +\infty)$.

Exercise 3 (Compositions)**.**

(a) $f'(x) = 87(2x+51)(x^2+51x+119)^{86}$, $x \in \mathbb{R}$.

(b) $f'(x) = (22x^2 - 49x - 42)x^2(x+2)^7(x-7)^{10}$, $x \in \mathbb{R}$.

(c) $f'(x) = \frac{2x+1}{x^2+x+2}$, $x \in \mathbb{R}$.

(d) $f'(x) = 6\sin(3x)\cos(3x) + \frac{3\log^2(2x)}{x}$, $x > 0$.

(e) $f'(x) = \frac{2}{x^3}e^{-\frac{1}{x^2}}$, $x \in (-\infty, 0) \cup (0, +\infty)$.

(f) $f'(x) = 2\sin x \cos x - 2x\cos(x^2)$, $x \in \mathbb{R}$.

(g) $f'(x) = \frac{1}{(1+x^2)\arctan x}$, $x > 0$.

(h) $f'(x) = \frac{8x}{x^4-1}$, $x \in (-\infty, -1) \cup (1, +\infty)$.

(i) $f'(x) = \frac{1}{1+x} \cdot \frac{1}{2\sqrt{x}}$, $x > 0$.

(j) $f'(x) = -27(3x^2-1)(x^3-x+2)^8\cos^2(x^3-x+2)^9\sin(x^3-x+2)^9\cos^3(x^3-x+2)^9$, $x \in \mathbb{R}$.

(k) $f'(x) = \frac{9}{\sqrt{(9-x^2)^3}}$, $x \in (-3, 3)$.

(l) $f'(x) = -\frac{2x(2x^2+2x-1)(1-x)^2}{(1+x)^2}$, $x \neq -1$.

(m) $f'(x) = -\frac{1}{x^2+3}$, $x \neq 0$.

(n) $f'(x) = \arcsin\sqrt{x} + \frac{\sqrt{x}}{2\sqrt{1-x}}$, $x \in (0, 1)$.

(o) $f'(x) = \cos x \cdot \cos(\sin x) \cdot \cos(\sin(\sin x))$, $x \in \mathbb{R}$.

(p) $f'(x) = \frac{6\log(\log^3 x)}{x\log x}$, $x > 1$.

(q) $f'(x) = (\log x + 1)x^x$, $x > 0$.

(r) $f'(x) = -\left(\frac{1}{x}\right)^{\frac{1}{x}} \cdot \frac{\log\frac{1}{x}+1}{x^2}$, $x > 0$.

(s) $f'(x) = (\sin x)^{\cos x}\left(\frac{\cos^2 x}{\sin x} - \sin x \cdot \log(\sin x)\right)$, $x \in (2k\pi, \pi + 2k\pi)$, $k \in \mathbb{Z}$.

(t) $f'(x) = \arccos\frac{1}{x^2+1} + \frac{2x(x+1)}{(x^2+1)^2\sqrt{1-\frac{1}{(1+x^2)^2}}}$, $x \neq 0$.

(u) $f'(x) = 2xe^{x^2-1} - \frac{1}{e^{2x}+1}$, $x \in \mathbb{R}$.

Exercise 4 (One-sided derivatives)**.**

(a) $f'(x) = \begin{cases} 1, & x \leq 0 \\ \frac{1}{1+x}, & x > 0 \end{cases}$.

(b) $f'(x) = \begin{cases} -1, & x < 1 \\ 2x-3, & x \in [1,2] \\ 1, & x > 2 \end{cases}$.

(c) $f(x) = \begin{cases} -2x, & x \in (0,1) \\ 2(x-1), & x \in (-\infty, 0) \cup (1, +\infty) \\ \text{Does not exist}, & x \in \{0, 1\} \end{cases}$.

(d) $f(x) = \begin{cases} 2x, & x > 0 \\ 0, & x = 0 \\ -2x, & x < 0 \end{cases}$.

(e) $f'(x) = \begin{cases} (x-2)e^{(x-2)^2}, & x \in (-\infty, -2) \cup (2, +\infty) \\ -(x-2)e^{(x-2)^2}, & x \in (-2, 2) \\ 0, & x = 2 \\ \text{Does not exist}, & x = -2 \end{cases}.$

(f) $f'(x) = \begin{cases} \frac{1}{x}, & x \in (-\infty, -e) \cup (e, +\infty) \\ -\frac{1}{x}, & x \in (-e, 0) \cup (0, e) \\ \text{Does not exist}, & x = \pm e \end{cases}.$

(g) $f'(x) = \begin{cases} \arcsin(x^2-1), & x \in (-\sqrt{2}, 0) \cup (0, \sqrt{2}) \\ \text{Does not exist}, & x = 0 \\ -\infty, & x = -\sqrt{2}_+ \\ +\infty, & x = \sqrt{2}_- \end{cases}.$

(h) $f'(x) = \begin{cases} -\sin x, & x \in (0, 1) \\ -2x\sin(x^2), & x \in (-\infty, 0) \cup (1, +\infty) \\ 0, & x = 0 \\ \text{Does not exist}, & x = 1 \end{cases}.$

Exercise 5 (L'Hôpital's rule)**.**

(a) $\frac{1}{2}$.

(b) 0.

(c) $-\infty$.

(d) $\frac{1}{2}$.

(e) $\frac{1}{5}$.

(f) $+\infty$.

(g) $-\frac{1}{2}$.

(h) -1.

(i) $\frac{6}{5}$.

(j) $+\infty$.

(k) 2.

(l) $-\frac{e}{2}$.

(m) $\frac{2}{3}$.

(n) 4.

(o) 0.

(p) 1.

(q) 0.

(r) 1.

(s) $\frac{3}{2}$.

(t) $\frac{3}{2}$.

(u) $\frac{\log 3}{\log 2}$.