V. Limits of functions

Theory.

Definition (Limit of a function)**.** We say that $A \in \mathbb{R}^*$ is a limit of a function f at $c \in \mathbb{R}^*$ if

$$\forall \varepsilon > 0\ \exists \delta > 0\ \forall x \in P(c, \delta) : f(x) \in B(A, \varepsilon).$$

If this is the case, we write $\lim_{x\to c} f(x) = A$.

For $c \in \mathbb{R}$, $P(c, \delta)$ represents the set $(c - \delta, c + \delta) \setminus \{c\}$, i.e. (δ)-punctured neighbourhood of $c \in \mathbb{R}$ and similarly, $B(c, \delta)$ is the set $(c - \delta, c + \delta)$, i.e. (δ)-neighbourhood of $c \in \mathbb{R}$. For $c = +\infty$ we have $P(+\infty, \delta) = B(+\infty, \delta) = (\frac{1}{\delta}, +\infty)$. Analogously also for $c = -\infty$.

Definition (Continuity)**.** We say that a function f is continuous at a point $c \in \mathbb{R}$ if

$$\lim_{x\to c} f(x) = f(c).$$

A function f is continuous on a non-degenerate interval I if it is continous at every inner point I.

Remark. *The limit is uniquely determined (if it exists). We also introduce the left and right (punctured) neighbourhood, which is then used in the analogous definition of a limit from the left or right of f at c. We also work with continuity at c from the left and right. There holds that*

$$\lim_{x\to c} f(x) = A \Leftrightarrow \left(\lim_{x\to c_-} f(x) = A \wedge \lim_{x\to c_+} f(x) = A \right).$$

Remark. *Just like in the chapter about sequences, we also have results talking about arithmetics of limits and about limits and inequalities, which also gives two policemen theorem.*

Claim. Constant functions, rational powers of x, exponential functions, logarithms, trigonometric and inverse trigonometric functions are continuous on their domains.

Claim. The sum, product, and composition of continuous functions is continuous.

Claim (Limit of a composition)**.** Let $c, A, B \in \mathbb{R}^*$ and functions f and g satisfy $\lim_{x\to c} g(x) = A$ and $\lim_{y\to A} f(y) = B$. Assume that at least one of the following conditions is satisfied:

(I) $\exists \eta > 0\ \forall x \in P(c, \eta) : g(x) \neq A$.

(C) The function f is continuous at A.

Then $\lim_{x\to c} f(g(x)) = B$.

Remark. *The analogous theorem holds also for the limit from the left and right.*

Claim (Known limits)**.** From the lecture we know (or will know) that

$$\lim_{x\to 0} \frac{\sin x}{x} = 1, \qquad \lim_{x\to 0} \frac{e^x - 1}{x} = 1, \qquad \lim_{x\to 1} \frac{\log x}{x - 1} = \lim_{x\to 0} \frac{\log(1 + x)}{x} = 1.$$

A simple corollary is

$$\lim_{x\to 0} \frac{\tan x}{x} = 1, \qquad \lim_{x\to 0} \frac{\arcsin x}{x} = 1, \qquad \lim_{x\to 0} \frac{\arctan x}{x} = 1, \qquad \lim_{x\to 0} \frac{1 - \cos x}{x^2} = \frac{1}{2}.$$

We also have the scale in the following form

$$\lim_{x\to+\infty} \frac{x^\beta}{a^x} = 0 \quad \text{and} \quad \lim_{x\to+\infty} \frac{\log^\alpha x}{x^\beta} = 0, \text{ where } \alpha, \beta > 0 \text{ and } a > 1.$$

Remark. *We also know limits of elementary functions, in particular, the following is true*

$$\lim_{x\to+\infty} x^\alpha = +\infty, \lim_{x\to+\infty} \frac{1}{x^\alpha} = 0 \textit{ where } \alpha > 0,$$
$$\lim_{x\to+\infty} e^x = +\infty, \lim_{x\to-\infty} e^x = 0, \lim_{x\to+\infty} \log x = +\infty, \lim_{x\to 0_+} \log x = -\infty,$$
$$\lim_{x\to \frac{k\pi}{2}_-} \tan x = +\infty, \lim_{x\to \frac{k\pi}{2}_+} \tan x = -\infty, \lim_{x\to+\infty} \arctan x = \frac{\pi}{2}, \lim_{x\to-\infty} \arctan x = -\frac{\pi}{2}.$$

Definition. General exponentiation is defined by $a^b = e^{b \log a}$ for $a > 0$ and $b \in \mathbb{R}$.

Exercise 1 (Elementary)**.** Sketch the graph and decide about the limit:

(a) $\lim_{x\to+\infty} x^3$.
(b) $\lim_{x\to 0} x$.
(c) $\lim_{x\to 3} x^2$.
(d) $\lim_{x\to 0_-} \frac{1}{x}$.
(e) $\lim_{x\to 0} \frac{1}{x}$.
(f) $\lim_{x\to 0} \frac{1}{x^2}$.
(g) $\lim_{x\to 0_+} \sqrt{x}$.
(h) $\lim_{x\to 0} \sqrt[3]{x}$.
(i) $\lim_{x\to -2_+} \sqrt[4]{x}$.
(j) $\lim_{x\to \frac{\pi}{2}} \tan x$.
(k) $\lim_{x\to -1_+} \arcsin x$.
(l) $\lim_{x\to 3\pi} \cos x$.

Exercise 2 (Continuity & AL)**.** Find limits:

(a) $\lim_{x\to 6}(x-2)^{11}$.
(b) $\lim_{x\to -\infty}(x+2)^2$.
(c) $\lim_{x\to 2} \log(x-3)-x$.
(d) $\lim_{x\to 0} \frac{2}{8+x}+\cos(\pi x)$.
(e) $\lim_{x\to+\infty} e^{4x}+\sqrt[4]{x}$.
(f) $\lim_{x\to+\infty} \frac{1}{\log x+1}$.
(g) $\lim_{x\to-\infty} \frac{4}{7+x}$.
(h) $\lim_{x\to\frac{\pi}{4}} \tan x$.
(i) $\lim_{x\to\sqrt{3}} \arctan x$.
(j) $\lim_{x\to\sqrt{2}-1} \arcsin \frac{x+1}{2}$.
(k) $\lim_{x\to-\infty} e^{-x}$.
(l) $\lim_{x\to 5} |-e^{-\frac{1}{x}}|+\log(x+1)$.
(m) $\lim_{x\to+\infty} e^{\frac{1}{3-x}}$.
(n) $\lim_{x\to+\infty} \sqrt{\arctan x+\frac{1}{x}}$.
(o) $\lim_{x\to+\infty} 2^{\sqrt{x^2+4x}-x}$.

Exercise 3 (Rational functions)**.** Find limits:

(a) $\lim_{x\to+\infty} \frac{3}{x^2+2}$.
(b) $\lim_{x\to+\infty} \frac{2x^4-x^2+x+1}{x^3-x^2+2}$.
(c) $\lim_{x\to+\infty} \frac{3^x-2^x+x^2}{x^6-2\cdot 3^x+1}$.
(d) $\lim_{x\to+\infty} \frac{e^x-\frac{1}{x}+e^{-x}}{e^x+\sin x}$.
(e) $\lim_{x\to-\infty} \frac{8x^3+3x+5-\frac{1}{x}}{x^3+4x^2-3}$.
(f) $\lim_{x\to 2} \frac{2x^3-x^2+x+1}{(x-2)^2+2}$.
(g) $\lim_{x\to -7} \frac{1}{7+x}$.
(h) $\lim_{x\to 1} \frac{x^2+4x-5}{x-1}$.
(i) $\lim_{x\to 1} \frac{x^2+4x-5}{(x-1)^2}$.
(j) $\lim_{x\to 0} \frac{x^3-2x}{2x^3+x^2-3x}$.
(k) $\lim_{x\to 2} \frac{(x^2-x-2)^{20}}{(x^3-12x+16)^{10}}$.
(l) $\lim_{x\to 0} \frac{(1+x)(1+2x)(1+3x)-1}{x}$.

Exercise 4 (Roots)**.** Find limits:

(a) $\lim_{x\to+\infty} \sqrt{\frac{x^2-2x-\frac{1}{x}}{3x^2+3}}$.
(b) $\lim_{x\to+\infty} \frac{\sqrt{x^2+1+\log^3 x}}{x+2}$.
(c) $\lim_{x\to+\infty} \sqrt{x+2}-\sqrt{x-1}$.
(d) $\lim_{x\to+\infty} x(\sqrt{x^2+1}-x)$.
(e) $\lim_{x\to+\infty} x^{\frac{1}{3}}[(x+2)^{\frac{2}{3}}-x^{\frac{2}{3}}]$.
(f) $\lim_{x\to-\infty} \frac{\sqrt{x^2+1}}{x}$.
(g) $\lim_{x\to 1_+} \frac{x}{\sqrt{x^2-1}}$.
(h) $\lim_{x\to -2_-} \frac{x+2}{\sqrt{(x+2)^2(x+4)}}$.
(i) $\lim_{x\to 0} \frac{\sqrt{x+4}-\sqrt{4}}{x}$.
(j) $\lim_{x\to 4} \frac{\sqrt{1+2x}-3}{\sqrt{x}-2}$.
(k) $\lim_{x\to-\infty} 2x(\sqrt{x^2+1}+x)$.
(l) $\lim_{x\to 3_+} \frac{x-3}{\sqrt{x^2-2-\sqrt{2x+1}}}$.
(m) $\lim_{x\to 1} \frac{\sqrt{2^{x-1}+1}-\sqrt{2^{x-1}+x}}{x^2-1}$.
(n) $\lim_{x\to 0} \frac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt[3]{1+x}-\sqrt[3]{1-x}}$.
(o) $\lim_{x\to 0_+} \frac{\sqrt[3]{\log\frac{1}{x}+x}-\sqrt[3]{\log\frac{1}{x}}}{\sqrt[3]{\log x}}$.

Exercise 5 (Exp and log functions)**.** Find limits:

(a) $\lim_{x\to 0} \frac{1-e^x}{x+1}$.
(b) $\lim_{x\to 0} \frac{\log(1+3x)}{x}$.
(c) $\lim_{x\to 0} \frac{1-e^{x^3}}{6x^3}$.
(d) $\lim_{x\to 0} \frac{\log(1+x^2)}{\log(1-x^2)}$.
(e) $\lim_{x\to 0} \frac{e^{2x}-1}{\log(1+5x)}$.
(f) $\lim_{x\to+\infty} x\log\left(1-\frac{3}{x}\right)$.
(g) $\lim_{x\to-\infty} \frac{\log(1+3^x)}{\log(1+2^x)}$.
(h) $\lim_{x\to 2} \frac{e-e^{x-1}}{2x-4}$.
(i) $\lim_{x\to+\infty} \log(x+1)-\log x$.

Exercise 6 (Trigonometric functions)**.** Find limits:

(a) $\lim_{x\to 3} \frac{\sin x}{x-3}$.
(b) $\lim_{x\to 0} \frac{\sin 5x}{x}$.
(c) $\lim_{x\to 4} \frac{\sin 3(x-4)^2}{(x-4)^2}$.
(d) $\lim_{x\to 2} \frac{\arcsin(x-2)}{\sin\frac{x-2}{x+2}}$.
(e) $\lim_{x\to 0_+} \frac{\tan\sqrt{x}}{\sqrt{8x}}$.
(f) $\lim_{x\to 0} \frac{x^4}{1-\cos 4x^2}$.
(g) $\lim_{x\to 0} \frac{\tan x-\sin x}{\sin^3 x}$.
(h) $\lim_{x\to 0} \frac{\sin 5x-\sin 3x}{\sin x}$.
(i) $\lim_{x\to 0} \frac{\arcsin x}{\sin(\arctan x)}$.
(j) $\lim_{x\to 1} \frac{\arctan(x-1)^2}{1-\cos(x-1)}$.
(k) $\lim_{x\to 0} \frac{\sqrt{1-\cos x^2}}{\cos x-1}$.
(l) $\lim_{x\to 1} \frac{\arcsin(\sqrt{x+8}-3)}{\arctan(x-1)}$.

Exercise 7 (Mix)**.** Find limits:

(a) $\lim_{x\to 0} \frac{\sqrt{x\sin x}}{e^{x^2}-1}$.

(b) $\lim_{x\to 0} \log \frac{x}{\sin x}$.

(c) $\lim_{x\to 0} \frac{1-e^{\sin x}}{x}$.

(d) $\lim_{x\to 0} \frac{\log\cos x}{x^2}$.

(e) $\lim_{x\to +\infty} \frac{\log(x^3-\arctan x)}{\log(x^2+\arctan x)}$.

(f) $\lim_{x\to 0} x\frac{\log(1+\sin^2 x)}{\log(1-x^3)}$.

(g) $\lim_{x\to +\infty} \frac{\sqrt{\log(x^2+4)-\log(x^2)}}{\arctan \frac{1}{x}}$.

(h) $\lim_{x\to 0} \frac{\sqrt{1+x\sin x}-1}{e^{x^2}-1}$.

(i) $\lim_{x\to 0} \frac{e^{x^4}-\cos x^2}{\arcsin(2x)^4}$.

(j) $\lim_{x\to 2_+} \frac{e-e^{x^2-3}}{\cos\sqrt{x-2}-1}$.

(k) $\lim_{x\to +\infty} \log(1+2^x)\cdot\log\left(1+\frac{3}{x}\right)$.

(l) $\lim_{x\to 1_-} \frac{\sqrt{e^2-e^{2x}}}{\log(1+\sqrt{2-2x})}$.

(m) $\lim_{x\to 0} \frac{1-e^{x^2}}{(\sin x+\arctan x)^2}$.

(n) $\lim_{x\to 0} \frac{\cos x-\cos 3x}{x^2}$.

(o) $\lim_{x\to 0} \frac{\sqrt{1+\tan x}-\sqrt{1+\sin x}}{x^3}$.

(p) $\lim_{x\to 0} \frac{x^2}{\sqrt{1+x\cdot\sin x}-\sqrt{\cos x}}$.

(q) $\lim_{x\to +\infty} \frac{\frac{1}{\sqrt{x^5}}}{\arcsin(\sqrt{x^5+1}-\sqrt{x^5-1})}$.

(r) $\lim_{x\to +\infty} \frac{1}{x}\cdot\left(\frac{1}{1-e^{\frac{3}{\sqrt{x}}}}\right)^2$.

(s) $\lim_{x\to 1_+} \frac{e^{(x+1)^2}-e^4}{\log(1+(x-1)^2)}$.

(t) $\lim_{x\to +\infty} \frac{\log(1-x+x^2)}{\log(x^{10}+x+1)}$.

(u) $\lim_{x\to \infty} \frac{\log(1+3^x)}{\log(1+2^x)}$.

(v) $\lim_{x\to +\infty} \frac{\log(1+4^x)}{x^2\log\left(1+\frac{4}{x}\right)}$.

Exercise 8 (Exponential trick)**.** Find limits:

(a) $\lim_{x\to 0} \left(\frac{1+x}{2+x}\right)^{(1-x)(1-\sqrt{x})}$.

(b) $\lim_{x\to +\infty} \left(1+\frac{1}{x}\right)^x$.

(c) $\lim_{x\to +\infty} \left(1-\frac{2}{x}\right)^x$.

(d) $\lim_{x\to 1} \left(\frac{1+x}{2+x}\right)^{\frac{1-x}{1-x^2}}$.

(e) $\lim_{x\to +\infty} \left(\frac{x+2}{2x+1}\right)^{x^2}$.

(f) $\lim_{x\to +\infty} \left(\frac{1+\log x}{\log x}\right)^{\log x}$.

(g) $\lim_{x\to 0} \left(1+x^2\right)^{\frac{1}{\tan^2 x}}$.

(h) $\lim_{x\to 0} \left(1+\tan x^2\right)^{\frac{1}{\sin^2 x}}$.

(i) $\lim_{x\to 0} \left(\frac{1+\tan x}{1+\sin x}\right)^{\frac{1}{\sin x}}$.

(j) $\lim_{x\to 0} \left(\frac{1+x\cdot 2^x}{1+x\cdot 3^x}\right)^{\frac{1}{x^2}}$.

(k) $\lim_{x\to -\infty}(1+5^{-x})^{\frac{1}{x}}$.

(l) $\lim_{x\to 0_+}(1+\sqrt{\arctan x})^{-\frac{1}{\log(1+\sqrt{x})}}$.

(m) $\lim_{x\to 0}(\cos 2x^2)^{\frac{1}{\arcsin^4 x}}$.

(n) $\lim_{x\to 0}(\log(2-e^x))^{\frac{\sqrt{1-3\sin x^2}-\sqrt{1+3\sin x^2}}{x^3}}$.

(o) $\lim_{x\to 0}(1+\sin^4\sqrt{x})^{\frac{1}{\cos x-e^{x^2}}}$.

(p) $\lim_{x\to +\infty}(1+e^{-x^3})^{\sqrt{x^2+5x}-\sqrt{x^2-3x}}$.

Results - V. Limits of functions

Exercise 1 (Elementary)**.**

(a) $+\infty$.
(b) 0.
(c) 9.
(d) $-\infty$.
(e) Does not exist.
(f) $+\infty$.
(g) 0.
(h) 0.
(i) Does not exist.
(j) Does not exist.
(k) $-\frac{\pi}{2}$.
(l) -1.

Exercise 2 (Continuity & AL)**.**

(a) 4^{11}.
(b) $+\infty$.
(c) Does not exist.
(d) $\frac{5}{4}$.
(e) $+\infty$.
(f) 0.
(g) 0.
(h) 1.
(i) $\frac{\pi}{3}$.
(j) $\frac{\pi}{4}$.
(k) $+\infty$.
(l) $\frac{1}{\sqrt[5]{e}} + \log 6$.
(m) 1.
(n) $\sqrt{\frac{\pi}{2}}$.
(o) 4.

Exercise 3 (Rational functions)**.**

(a) 0.
(b) $+\infty$.
(c) $-\frac{1}{2}$.
(d) 1.
(e) -8.
(f) $\frac{15}{2}$.
(g) Does not exist.
(h) 6.
(i) Does not exist.
(j) $\frac{2}{3}$.
(k) $\left(\frac{3}{2}\right)^{10}$.
(l) 6.

Exercise 4 (Roots)**.**

(a) $\frac{1}{\sqrt{3}}$.
(b) 1.
(c) 0.
(d) $\frac{1}{2}$.
(e) $\frac{4}{3}$.
(f) -1.
(g) $+\infty$.
(h) $-\frac{1}{\sqrt{2}}$.
(i) $\frac{1}{4}$.
(j) $\frac{4}{3}$.
(k) -1.
(l) $\frac{\sqrt{7}}{2}$.
(m) $-\frac{1}{4\sqrt{2}}$.
(n) $\frac{3}{2}$.
(o) 0.

Exercise 5 (Exp and log functions)**.**

(a) 0.
(b) 3.
(c) $-\frac{1}{6}$.
(d) -1.
(e) $\frac{2}{5}$.
(f) -3.
(g) 0.
(h) $-\frac{e}{2}$.
(i) 0.

Exercise 6 (Trigonometric functions)**.**

(a) Neexistuje.
(b) 5.
(c) 3.
(d) 4.
(e) $\frac{\sqrt{2}}{4}$.
(f) $\frac{1}{8}$.
(g) $\frac{1}{2}$.
(h) 2.
(i) 1.
(j) 2.
(k) $-\sqrt{2}$.
(l) $\frac{1}{6}$.

Exercise 7 (Mix)**.** Find limits:

(a) $+\infty$.

(b) 0.

(c) -1.

(d) $-\frac{1}{2}$.

(e) $\frac{3}{2}$.

(f) -1.

(g) 2.

(h) $\frac{1}{2}$.

(i) $\frac{3}{32}$.

(j) $8e$.

(k) $3\log 2$.

(l) e.

(m) $-\frac{1}{4}$.

(n) 4.

(o) $\frac{1}{4}$.

(p) $\frac{4}{3}$.

(q) 1.

(r) $\frac{1}{9}$.

(s) $+\infty$.

(t) $\frac{1}{5}$.

(u) $\frac{\log 3}{\log 2}$.

(v) $\frac{\log 2}{2}$.

Exercise 8 (Exponential trick)**.**

(a) $\frac{1}{2}$.

(b) e.

(c) $\frac{1}{e^2}$.

(d) $\sqrt{\frac{2}{3}}$.

(e) 0.

(f) e.

(g) e.

(h) e.

(i) 1.

(j) $\frac{2}{3}$.

(k) $\frac{1}{5}$.

(l) $\frac{1}{e}$.

(m) $\frac{1}{e^2}$.

(n) e^3.

(o) $\frac{1}{e^{\frac{2}{3}}}$.

(p) 1.