Mathematics I - Midterm

You have 60 minutes and can use any literature (notes, tables, textbooks...), but no technical devices (phones, calculators, watches...).

Please, do not forget to mention AL and indicate for which sequences you use Scale. Also, explain an application of RL. Finally, either Two policemen theorem or the theorem about the product of bounded and vanishing sequences might be handy.

Good luck!

Exercise 1. [8pts] Find the limit:

$$\lim_{n\to\infty} \frac{\sqrt{n^4+4n}-\sqrt{n^4+n}}{\sqrt{2n^2+1}-\sqrt{2n^2-1}}.$$

Exercise 2. [7pts] Find the limit:

$$\lim_{n\to\infty} \frac{3^n+8n^n-\log^4 n+2n}{n^6-n\cdot n^{n-1}-\sin n+n!}.$$

Midterm - Solution

Solution to Exercise 1. For the difference of square roots in the denominator we get

$$\begin{aligned}\frac{1}{\sqrt{2n^2+1}-\sqrt{2n^2-1}} &= \frac{1}{\sqrt{2n^2+1}-\sqrt{2n^2-1}} \cdot \frac{\sqrt{2n^2+1}+\sqrt{2n^2-1}}{\sqrt{2n^2+1}+\sqrt{2n^2-1}} \text{ [0,5pt]} \\ &= \sqrt{n^2} \cdot \frac{\sqrt{2+\frac{1}{n^2}}+\sqrt{2-\frac{1}{n^2}}}{(2n^2+1)-(2n^2-1)} \text{ [1,5pt]} \\ &= \frac{n}{2} \cdot \left(\sqrt{2+\frac{1}{n^2}}+\sqrt{2-\frac{1}{n^2}}\right) \text{ [0,5pt]}\end{aligned}$$

and for roots in the numerator we have

$$\begin{aligned}\sqrt{n^4+4n}-\sqrt{n^4+n} &= (\sqrt{n^4+4n}-\sqrt{n^4+n}) \cdot \frac{\sqrt{n^4+4n}+\sqrt{n^4+n}}{\sqrt{n^4+4n}+\sqrt{n^4+n}} \text{ [0,5pt]} \\ &= \frac{1}{\sqrt{n^4}} \cdot \frac{(n^4+4n)-(n^4+n)}{\sqrt{1+\frac{5}{n^3}}+\sqrt{1+\frac{1}{n^3}}} \text{ [1,5pt]} \\ &= \frac{3}{n} \cdot \frac{1}{\sqrt{1+\frac{5}{n^3}}+\sqrt{1+\frac{1}{n^3}}}. \text{ [0,5pt]}\end{aligned}$$

Together,

$$\lim_{n\to\infty} \frac{\sqrt{n^4+4n}-\sqrt{n^4+n}}{\sqrt{2n^2+1}-\sqrt{2n^2-1}} = \lim_{n\to\infty} \frac{n}{2} \cdot \frac{3}{n} \cdot \frac{\sqrt{2+\frac{1}{n^2}}+\sqrt{2-\frac{1}{n^2}}}{\sqrt{1+\frac{5}{n^3}}+\sqrt{1+\frac{1}{n^3}}} \underset{\text{[0,5pt]}}{\overset{AL}{=}} \frac{3}{2} \cdot \frac{\sqrt{2}+\sqrt{2}}{1+1} = \frac{3\sqrt{2}}{2}. \text{ [1pt]}$$

In the calculation we used that due to AL we have, e.g.

$$2+\frac{1}{n^2} \to 2 \text{ as } n \to +\infty \text{ [0,5pt]}$$

and together with RL it implies

$$\sqrt{2+\frac{1}{n^2}} \to \sqrt{2} \text{ as } n \to +\infty. \text{ [0,5pt]}$$

Convergence of other roots is analogous. [0,5pt]

Solution to Exercise 2. We have

$$\begin{aligned}\lim_{n\to\infty} \frac{3^n+8n^n-\log^4 n+2n}{n^6-\underbrace{n\cdot n^{n-1}}_{=n^n \text{ [0,5pt]}}-\sin n+n!} &= \lim_{n\to\infty} \frac{n^n}{n^n} \cdot \frac{\frac{3^n}{n^n}+8-\frac{\log^4 n}{n^n}+\frac{n}{n^n}}{\frac{n^6}{n^n}-1-\frac{\sin n}{n^n}+\frac{n!}{n^n}} \text{ [1,5pt]} \\ &= \lim_{n\to\infty} \frac{\frac{3^n}{n^n}+8-\frac{\log^4 n}{n^n}+\frac{1}{n^{n-1}}}{\frac{1}{n^{n-6}}-1-\frac{\sin n}{n^n}+\frac{n!}{n^n}} \text{ [1pt]} \underset{\text{[0,5pt]}}{\overset{AL}{=}} \frac{0+8-0+0}{0-1-0+0} = -8. \text{ [1pt]}\end{aligned}$$

In the calculation we used two ingredients. Due to Scale we know that

$$\frac{3^n}{n^n} \to 0 \text{ [0,5pt]}, \quad \frac{\log^4 n}{n^n} \to 0 \text{ [0,5pt]}, \quad \frac{n!}{n^n} \to 0 \text{ [0,5pt]}.$$

Further, due to either Two policemen theorem or product of bounded and vanishing sequences we have

$$\frac{\sin n}{n^n} \to 0 \text{ as } n \to +\infty \text{ [1pt]}.$$

Limits of $\frac{1}{n^{n-1}}$, $\frac{1}{n^{n-6}}$, $\frac{3}{n^{2n}}$ are clearly 0 (we do not need to comment it).