IV. Limits of sequences

Theory.

Definition (Limit of a sequence)**.** Let $\{a_n\}_{n=1}^\infty$ be a sequence of real numbers and $A \in \mathbb{R}$. We say that A is a limit of $\{a_n\}_{n=1}^\infty$ (and then we write $A = \lim_{n\to\infty} a_n$) if

$$\forall \varepsilon > 0 \, \exists n_0 \in \mathbb{N} \, \forall n \geq n_0, n \in \mathbb{N} : |a_n - A| < \varepsilon.$$

The sequence $\{a_n\}_{n=1}^\infty$ has a limit $+\infty$ [or $-\infty$], if

$$\forall K \in \mathbb{R} \, \exists n_0 \in \mathbb{N} \, \forall n \geq n_0, n \in \mathbb{N} : a_n > K \text{ [or } < K].$$

Remark. *We say that a sequence $\{a_n\}_{n=1}^\infty \subset \mathbb{R}$ is convergent if it has a limit $A \in \mathbb{R}$ (it converges to A). It is divergent if it has a limit $A \in \{-\infty, +\infty\}$ (it diverges to either $-\infty$ or $+\infty$).*

Claim (Uniqueness of a limit)**.** There exists at most one limit to a given sequence of real numbers.

Claim (Limit of a subsequence)**.** Let $\{a_n\}_{n=1}^\infty$ be a sequence of real numbers with a limit $A \in \mathbb{R}^*$. If $\{b_k\}_{k=1}^\infty$ is a subsequence of $\{a_n\}_{n=1}^\infty$, then $\lim_{k\to\infty} b_k = A$.

Claim (Arithmetics of limits)**.** Let us consider sequences $\{a_n\}_{n=1}^\infty$, $\{b_n\}_{n=1}^\infty \subset \mathbb{R}$ such that $\lim_{n\to\infty} a_n = A \in \mathbb{R}^*, \lim_{n\to\infty} b_n = B \in \mathbb{R}^*$. Then (if the RHS is well-defined)

$$\lim_{n\to\infty} (a_n \pm b_n) = A \pm B, \quad \lim_{n\to\infty} a_n \cdot b_n = A \cdot B, \quad \lim_{n\to\infty} \frac{a_n}{b_n} = \frac{A}{B}.$$

Claim (Two policemen / Sandwich)**.** Suppose that sequences $\{a_n\}_{n=1}^\infty$, $\{b_n\}_{n=1}^\infty$, $\{c_n\}_{n=1}^\infty \subset \mathbb{R}$ satisfy

$$\text{(i) } \exists n_0 \in \mathbb{N} \, \forall n \in \mathbb{N}, \, n \geq n_0 : a_n \leq c_n \leq b_n \quad \text{and} \quad \text{(ii) } \lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = A \in \mathbb{R}^*.$$

Then $\lim_{n\to\infty} c_n = A$.

Claim (Vanishing and bounded sequences)**.** Let us consider sequences $\{a_n\}_{n=1}^\infty$, $\{b_n\}_{n=1}^\infty \subset \mathbb{R}$ such that

$$\text{(i) } \lim_{n\to\infty} a_n = 0 \quad \text{and} \quad \text{(ii) } \{b_n\}_{n=1}^\infty \text{ is bounded.}$$

Then $\lim_{n\to\infty}(a_n \cdot b_n) = 0$.

Claim (Root law)**.** Let $\{a_n\}_{n=1}^\infty$ be a sequence of real numbers with a limit $A \in \mathbb{R}^*$ and $m \in \mathbb{N}$. Then

(i) $\lim_{n\to\infty} \sqrt[m]{a_n} = \sqrt[m]{\lim_{n\to\infty} a_n} = \sqrt[m]{A}$ if m is odd and

(ii) $\lim_{n\to\infty} \sqrt[m]{a_n} = \sqrt[m]{\lim_{n\to\infty} a_n} = \sqrt[m]{A}$ if m is even and $\exists n_0 > 0$ such that $a_n \geq 0$ for all $n \geq n_0$.

Claim (Scale: $\log^{\text{sth}} n \ll n^{\text{sth}} \ll \text{sth}^n \ll n! \ll n^n$)**.** There holds the following:

(i) $\lim_{n\to\infty} \frac{\log^\alpha n}{n^\beta} = 0$, if α, $\beta > 0$.

(ii) $\lim_{n\to\infty} \frac{n^\beta}{a^n} = 0$, if $a > 1$ and $\beta > 0$.

(iii) $\lim_{n\to\infty} \frac{a^n}{n!} = 0$, if $a > 0$.

(iv) $\lim_{n\to\infty} \frac{n!}{n^n} = 0$.

Remark. *We will work in $\mathbb{R}^*$ (extended real line). The following operations are not defined here:*

$$(+\infty) + (-\infty), \quad (-\infty) + (+\infty), \quad (+\infty) - (+\infty), \quad (-\infty) - (-\infty),$$
$$(\pm\infty) \cdot 0, \quad 0 \cdot (\pm\infty), \quad \frac{+\infty}{+\infty}, \quad \frac{-\infty}{-\infty}, \quad \frac{+\infty}{-\infty}, \quad \frac{-\infty}{+\infty}, \quad \frac{a}{0} \text{ for } a \in \mathbb{R}^*.$$

Remark. *Finally, we will use the following well-known formulas.*

$$a^2 - b^2 = (a-b)(a+b),$$
$$a^3 - b^3 = (a-b)(a^2 + ab + b^2),$$
$$(a+b)^n = \sum_{k=0}^n \binom{n}{k} a^{n-k} b^k, \text{ where } \binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

Exercise 1 (Elementary)**.** Sketch the graph, find the limit:

(a) $\lim_{n\to\infty} 2$.
(b) $\lim_{n\to\infty} n$.
(c) $\lim_{n\to\infty} \frac{1}{n}$.
(d) $\lim_{n\to\infty} e^n$.
(e) $\lim_{n\to\infty} \frac{n+7}{n}$.
(f) $\lim_{n\to\infty} n^3$.
(g) $\lim_{n\to\infty} \frac{1}{n^2}$.
(h) $\lim_{n\to\infty} e^{-n}$.
(i) $\lim_{n\to\infty} \left(\frac{5}{6}\right)^n$.
(j) $\lim_{n\to\infty} \sqrt{n}$.
(k) $\lim_{n\to\infty} \frac{1}{\sqrt{n}}$.
(l) $\lim_{n\to\infty} \log n$.
(m) $\lim_{n\to\infty} \arctan(-n)$.
(n) $\lim_{n\to\infty} n^n$.
(o) $\lim_{n\to\infty} n!$.
(p) $\lim_{n\to\infty} \sin\frac{1}{n}$.

Exercise 2 (Rational functions & roots)**.** Find limits:

(a) $\lim_{n\to\infty} -n^7+13n^4-123n+12$.
(b) $\lim_{n\to\infty} \frac{n^3+2n-6}{n+2}$.
(c) $\lim_{n\to\infty} \frac{n^4-2n^3+n}{5n^4+2n^2}$.
(d) $\lim_{n\to\infty} \frac{n^2-n}{2n^3-n^2+2}$.
(e) $\lim_{n\to\infty} \left(\frac{1+2+3+\cdots+n}{n+2}-\frac{n}{2}\right)$.
(f) $\lim_{n\to\infty} \frac{(n+4)^{100}-(n+3)^{100}}{(n+2)^{100}-n^{100}}$.
(g) $\lim_{n\to\infty} \frac{(n^2+2)^{10}-n^{20}}{(n+3)^{19}-n^{19}}$.
(h) $\lim_{n\to\infty} \frac{\sqrt{n^2+1}}{n-2}+\frac{\sqrt[5]{n^3}+1}{n-1}$.
(i) $\lim_{n\to\infty} \sqrt{n+11}+\sqrt{n}$.
(j) $\lim_{n\to\infty} \sqrt{n+11}-\sqrt{n}$.
(k) $\lim_{n\to\infty} \sqrt[3]{n+2}-\sqrt[3]{n}$.
(l) $\lim_{n\to\infty} \frac{\sqrt{n}(\sqrt{n-3}-\sqrt{n})}{\sqrt{n^2-1}-\sqrt{(n+2)^2}}$.
(m) $\lim_{n\to\infty} \frac{\sqrt{n+1}-1}{\sqrt{n^3+n^2}-\sqrt{n^3+1}}$.
(n) $\lim_{n\to\infty} \frac{\sqrt[3]{n^2+1}-\sqrt[3]{n^2-6n}}{\sqrt{n^2-1}-\sqrt{n^2+1}}$.
(o) $\lim_{n\to\infty} \sqrt[3]{\sqrt{n^7}+\sqrt[3]{n^7}}-\sqrt[3]{\sqrt{n^7}-\sqrt[3]{n^7}}$.
(p) $\lim_{n\to\infty} \frac{\sqrt[3]{n^2+6}-\sqrt[3]{n^2+4}}{\sqrt[3]{n^2+5}-\sqrt[3]{n^2+1}}$.

Exercise 3 (Scale)**.** Find limits:

(a) $\lim_{n\to\infty} \frac{1^n+2^n+3^n+4^n+5^n}{(5{,}0001)^n}$.
(b) $\lim_{n\to\infty} \frac{(-2)^n+3^n}{(-2)^{n+1}+3^{n+1}}$.
(c) $\lim_{n\to\infty} \frac{4^n+n^7+1}{n^8+n!}$.
(d) $\lim_{n\to\infty} \frac{\log^3 n+6n^2+n+\log n}{1+\log^2 n+n^2}$.
(e) $\lim_{n\to\infty} \frac{\log n(\sqrt{\log n}-\log n^4)}{\log^2 n^2+\log n+17}$.
(f) $\lim_{n\to\infty} \frac{\log^9 n+e^{2n+1}+n^4}{(e^n+n^2+2^n)^2}$.
(g) $\lim_{n\to\infty} \frac{3^n+n^5+(n+1)!}{n(n^6+n!)}$.
(h) $\lim_{n\to\infty} \frac{n^{n+1}+2^n+n^{19}}{n!+4(n+1)n^n}$.
(i) $\lim_{n\to\infty} \frac{\sqrt{n+3^n}+\sqrt{2^n+n^2}}{n^n}$.
(j) $\lim_{n\to\infty} \frac{(n+3)!-(n+1)!}{(n+3)!+(n+1)!}$.
(k) $\lim_{n\to\infty} \frac{\sqrt{3^n+n^3}-\sqrt{3^n+3}}{\sqrt{3^n+n^2}-\sqrt{3^n-n}}$.
(l) $\lim_{n\to\infty} \frac{n!}{n^{\frac{n}{2}}(\sqrt{n^n-2^n}-\sqrt{n!+n^n})}$.

Exercise 4 (Oscillations)**.** Find limits (if they exist):

(a) $\lim_{n\to\infty}(-1)^n$.
(b) $\lim_{n\to\infty} \frac{(-1)^n}{n}$.
(c) $\lim_{n\to\infty}(-1)^n n$.
(d) $\lim_{n\to\infty} \cos(\pi+2\pi n)$.
(e) $\lim_{n\to\infty} \left(-\frac{1}{3}\right)^n$.
(f) $\lim_{n\to\infty}(-2)^n$.
(g) $\lim_{n\to\infty} \sin(\pi n)$.
(h) $\lim_{n\to\infty} \sin\frac{\pi n}{2}$.
(i) $\lim_{n\to\infty} \frac{\sin n}{n}$.

Exercise 5 (Mix)**.** Find limits:

(a) $\lim_{n\to\infty} \cos(\pi n)\cdot n^{\frac{3}{4}}$.
(b) $\lim_{n\to\infty} \frac{1+2+3+\cdots+n}{n^2}$.
(c) $\lim_{n\to\infty} \frac{\sqrt{n!}(2^{2n}-3^n)}{\sqrt{(n+1)!+2^n}-\sqrt{n!+3^n}}$.
(d) $\lim_{n\to\infty} \frac{(n+3)^{30}-(n-3)^{30}}{(n+2)^{30}-(n-1)^{30}}$.
(e) $\lim_{n\to\infty} \sqrt{4^n+n^4}-\sqrt{4^n+2^n}$
(f) $\lim_{n\to\infty} \frac{(-3)^n-n^4+5^n}{7\cdot5^n+n^2-3}$.
(g) $\lim_{n\to\infty} \frac{(-1)^n}{n}+\frac{(-1)^n+1}{2}$.
(h) $\lim_{n\to\infty} \frac{\sqrt{3\cdot2^n+n}-\sqrt{2^{n+1}-5!}}{\sqrt{2^n+n}}$.
(i) $\lim_{n\to\infty} \frac{1}{\sqrt[6]{n}}\cdot\frac{\sqrt{n^3+n}-\sqrt{n^3-1}}{\sqrt[3]{n+1}-\sqrt[3]{n+2}}$.
(j) $\lim_{n\to\infty} \frac{\sqrt{(n+1)^{10}+1}-\sqrt{(n+4)^{10}+1}}{n^4+2n+3}$.
(k) $\lim_{n\to\infty} \frac{1^2+2^2+3^2+\cdots+n^2}{n^3}$.
(l) $\lim_{n\to\infty}(-1)^n\sqrt{n}(\sqrt{n+1}-\sqrt{n})$.
(m) $\lim_{n\to\infty} \frac{\sqrt{n^{2n}+n+n!}}{n^n+\sqrt{3^n+1}}$.
(n) $\lim_{n\to\infty} \frac{n^2+n+1}{\sqrt[3]{n^{3n}+n^{2n+2}+n^2}-\sqrt[3]{n^{3n}-n^4}}$.

Results - IV. Limits of sequences

Exercise 1 (Elementary)**.**

(a) 2.
(b) ∞.
(c) 0.
(d) ∞.
(e) 1.
(f) ∞.
(g) 0.
(h) 0.
(i) 0.
(j) ∞.
(k) 0.
(l) ∞.
(m) $-\frac{\pi}{2}$.
(n) ∞.
(o) ∞.
(p) 0.

Exercise 2 (Rational functions & roots)**.**

(a) $-\infty$.
(b) ∞.
(c) $\frac{1}{5}$.
(d) 0.
(e) $-\frac{1}{2}$.
(f) $\frac{1}{2}$.
(g) $\frac{20}{57}$.
(h) 1.
(i) ∞.
(j) 0.
(k) 0.
(l) $\frac{3}{4}$.
(m) 2.
(n) $-\infty$.
(o) $\frac{2}{3}$.
(p) $\frac{1}{2}$.

Exercise 3 (Scale)**.**

(a) 0.
(b) $\frac{1}{3}$.
(c) 0.
(d) 6.
(e) -1.
(f) e.
(g) 1.
(h) $\frac{1}{4}$.
(i) 0.
(j) 1.
(k) ∞.
(l) -2.

Exercise 4 (Oscillations)**.**

(a) The limit does not exist.
(b) 0.
(c) The limit does not exist.
(d) -1.
(e) 0.
(f) The limit does not exist.
(g) 0.
(h) The limit does not exist.
(i) 0.

Exercise 5 (Mix)**.**

(a) The limit does not exist.
(b) $\frac{1}{2}$.
(c) ∞.
(d) 2.
(e) $-\frac{1}{2}$.
(f) $\frac{1}{7}$.
(g) The limit does not exist.
(h) $\frac{1}{\sqrt{3}+\sqrt{2}} = \sqrt{3} - \sqrt{2}$.
(i) $-\frac{3}{2}$.
(j) -15.
(k) $\frac{1}{3}$.
(l) The limit does not exist.
(m) 1.
(n) 3.