III. Infimum and supremum

Theory.

Definition. Number $g \in \mathbb{R}$ is called **infimum** of the set $M \subset \mathbb{R}$ (we denote it by $\inf M$) provided that

(1) $\forall x \in M : x \geq g$ (= g is a lower bound of M) and

(2) $\forall g' > g \, \exists x \in M : x < g'$ (= g is the greatest lower bound).

If there exists a number from M satisfying (1), then (2) holds for free and we call this number **minimum** of the set M (we write $\min M$). In this case we say that infimum is attained. Similarly, number $G \in \mathbb{R}$ is called **supremum** of the set $M \subset \mathbb{R}$ (we write $\sup M$) if there holds

(3) $\forall x \in M : x \leq G$ (= G is an upper bound of M) a

(4) $\forall G' < G \; \exists x \in M : x > G'$ (= G is the smallest upper bound).

If there exists a number from M satisfying (3), then (4) holds for free and we call this number **maximum** of the set M (we write $\max M$); we say that supremum is attained.

Claim. Let $\emptyset \neq M \subset \mathbb{R}$ be a set bounded from below [above]. Then M has unique infimum [supremum].

Claim. There holds so-called **Archimedean property**, i.e.: $\forall x \in \mathbb{R} \, \exists n \in \mathbb{N} : n > x$.

Definition. If $M \subset \mathbb{R}$ is not bounded from below [above], then we define $\inf M = -\infty$ [$\sup M = +\infty$].

Exercise 1. In the following examples find infimum and supremum of a given set. Decide about the existence of its minimum and maximum.

(a) $A = \{8\}$.

(b) $B = \{5, 6\}$.

(c) $C = \{1, -9, 7, -3, 50\}$.

(d) $D = \{x \in \mathbb{R};\ x \leq 0\}$.

(e) $E = \{x \in \mathbb{R};\ x > 0\}$.

(f) $F = \langle -2, 5)$.

(g) $G = (-2, 0\rangle \cup \{1\} \cup (\langle 2, 4\rangle \cap (3, 4))$.

(h) $H = \{x \in \mathbb{R};\ x^2 < 16\}$.

(i) $I = \{x \in \mathbb{R};\ \frac{1}{x-1} \geq 2\}$.

(j) $J = \{x \in \mathbb{R};\ |x - 3| \leq 2\}$.

(k) $K = \{x \in \mathbb{R};\ ||x - 1| - |x - 2|| < 1\}$.

(l) $L = \{x \in \mathbb{R};\ \arctan x \geq \frac{\pi}{3}\}$.

(m) $M = \{\arctan x;\ x \geq -1\}$.

(n) $N = \{\sin x;\ x \in \langle 0, 2\pi)\}$.

(o) $O = \{\sin x;\ x \in (0, \pi)\}$.

(p) $P = \{x^2;\ x \in (-2, 3)\}$.

Exercise 2. In the following examples find infimum and supremum of a given set. Decide about the existence of its minimum and maximum.

(a) $A = \{\frac{1}{n};\ n \in \mathbb{N}\}$.

(b) $B = \{\frac{n}{n+1};\ n \in \mathbb{N}\}$.

(c) $C = \{2^{-n} + 3^{-n};\ n \in \mathbb{N}\}$.

(d) $D = \{2^{-n} + 3^{-n};\ n \in \mathbb{Z}\}$.

(e) $E = \{1 - \sum_{k=1}^{n} \frac{1}{3^k};\ n \in \mathbb{N}\}$.

(f) $F = \{n - 2\sum_{k=0}^{n} k;\ n \in \mathbb{N}\}$.

(g) $G = \{(-1)^n + \frac{1}{1+n};\ n \in \mathbb{N}\}$.

(h) $H = \{\frac{n}{n+m};\ n, m \in \mathbb{N}\}$.

(i) $I = \{n^2 - m^2;\ n, m \in \mathbb{N}\}$.

(j) $J = \{n^2 - m^2;\ n, m \in \mathbb{N}, n > m\}$.

(k) $K = \{n^2 - m^2;\ n, m \in \mathbb{N}, n \leq m\}$.

(l) $L = \{\cos\left(n + \frac{1}{n}\right)\pi;\ n \in \mathbb{N}\}$.

(m) $M = \{\cos\left(2n + \frac{1}{2n}\right)\pi;\ n \in \mathbb{N}\}$.

(n) $N = \{\tan \frac{\pi n}{2(n+m)};\ n \in \mathbb{N}_0, m \in \mathbb{N}\}$.

Exercise 3 (Hard)**.** Let $\emptyset \neq A \subset \mathbb{R}$ be a bounded set and $B := \{|x - y|; x, y \in A\}$. Show the following:

(a) B has supremum and infimum.

(b) There holds $\sup B = \sup A - \inf A$.

(c) Finally, $\inf B = 0$.

Results - III. Infimum and supremum

Exercise 1.

(a) $\inf A = \min A = \sup A = \max A = 8$.

(b) $\inf B = \min B = 5$ and $\sup B = \max B = 6$.

(c) $\inf C = \min C = -9$ and $\sup C = \max C = 50$.

(d) $\inf D = -\infty$, $\min D$ does not exist and $\sup D = \max D = 0$.

(e) $\inf E = 0$, $\min E$ does not exist and $\sup E = +\infty$, $\max E$ does not exist.

(f) $\inf F = \min F = -2$ and $\sup F = 5$, $\max F$ does not exist.

(g) $\inf G = -2$, $\min G$ does not exist $\sup G = 4$, $\max G$ does not exist.

(h) $\inf H = -4$, $\min H$ does not exist $\sup H = 4$, $\max H$ does not exist.

(i) $\inf I = 1$, $\min I$ does not exist $\sup I = \max I = \frac{3}{2}$.

(j) $\inf J = \min J = 1$ and $\sup J = \max J = 5$.

(k) $\inf K = 1$, $\min K$ does not exist $\sup K = 2$, $\max K$ does not exist.

(l) $\inf L = \min L = \sqrt{3}$ and $\sup L = +\infty$, $\max L$ does not exist.

(m) $\inf M = \min M = -\frac{\pi}{4}$ and $\sup M = +\infty$, $\max M$ does not exist.

(n) $\inf N = \min N = -1$ and $\sup N = \max N = 1$.

(o) $\inf O = 0$, $\min O$ does not exist and $\sup O = \max O = 1$.

(p) $\inf P = \min P = 0$ and $\sup P = 9$, $\max P$ does not exist.

Exercise 2.

(a) $\inf A = 0$, minium does not exist and $\sup A = \max A = 1$.

(b) $\inf B = \min B = \frac{1}{2}$ and $\sup B = 1$, maximum does not exist.

(c) $\inf C = 0$, minimum does not exist and $\sup C = \max C = \frac{5}{6}$.

(d) $\inf D = 0$, minimum does not exist and $\sup D = +\infty$, maximum does not exist.

(e) $\inf E = \frac{1}{2}$, minimum does not exist and $\sup E = \max E = \frac{2}{3}$.

(f) $\inf F = -\infty$, $\min F$ does not exist and $\sup F = \max F = -1$.

(g) $\inf G = -1$, minimum does not exist and $\sup G = \max G = \frac{4}{4}$.

(h) $\inf H = 0$, minimum does not exist and $\sup H = 1$, maximum does not exist.

(i) $\inf I = -\infty$, minimum does not exist and $\sup I = +\infty$, maximum does not exist.

(j) $\inf J = \min J = 3$ and $\sup J = +\infty$, maximum does not exist.

(k) $\inf K = -\infty$, minimum does not exist and $\sup K = \max K = 0$.

(l) $\inf L = -1$, minimum does not exist and $\sup L = \max L = 1$.

(m) $\inf M = \min M = 0$ and $\sup M = 1$, maximum does not exist.

(n) $\inf N = \min N = 0$ and $\sup N = +\infty$, maximum does not exist.