II. Mathematical proofs

Exercise 1. Prove directly the following well-known statements.

(a) $\sin\frac{\pi}{6} = \frac{1}{2} = \cos\frac{\pi}{3}$, $\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2} = \cos\frac{\pi}{6}$ and $\sin\frac{\pi}{4} = \frac{\sqrt{2}}{2} = \cos\frac{\pi}{4}$.

(b) $x^2 + y^2 \geq 2xy$ for all $x, y \in \mathbb{R}$. [special case of AM-GM inequality]

(c) $\sum_{k=0}^{n} q^k = \frac{1-q^{n+1}}{1-q}$ for any $q \in \mathbb{R} \setminus \{1\}$ and any $n \in \mathbb{N}_0$.

(d) $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$ for any $n \in \mathbb{N}$.

(e) Quadratic formula.

(f) $\sqrt{x}$ is increasing function on $[0, +\infty)$.

(g) $\sqrt[3]{x}$ is increasing function on $\mathbb{R}$.

(h) $|x + y| \leq |x| + |y|$ for all $x, y \in \mathbb{R}$. [Triangle inequality]

(i) Pythagorean theorem.

Exercise 2. Use contradiction to prove each of the following statements.

(a) $\sqrt{5}$ is an irrational number.

(b) If $a, b \in \mathbb{Z}$ such that $a + b \geq 20$, then either $a \geq 10$ or $b \geq 10$.

(c) If $x \neq y$ are positive reals, then $\frac{x}{y} + \frac{y}{x} > 2$.

(d) For all positive reals x, y there holds $\frac{2}{x} + \frac{2}{y} \neq \frac{4}{x+y}$.

(e) If k^2 is even, then k is even.

(f) If k and l are odd integers, then there does not exist an integer m such that $k^2 + l^2 = m^2$.

Exercise 3. Prove each of the following statements using mathematical induction.

(a) $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$ for any $n \in \mathbb{N}$.

(b) $\sum_{k=0}^{n} q^k = \frac{1-q^{n+1}}{1-q}$ for any $q \in \mathbb{R} \setminus \{1\}$ and any $n \in \mathbb{N}_0$.

(c) The sum of the first n odd numbers is equal to n^2.

(d) $\sum_{k=1}^{n} k^2 = \frac{1}{6}n(n+1)(2n+1)$ for any $n \in \mathbb{N}$.

(e) $\sum_{k=1}^{n} k^3 = \left(\sum_{k=1}^{n} k\right)^2$ for any $n \in \mathbb{N}$.

(f) $\sum_{k=1}^{n} k(k-1) = \frac{n(n^2-1)}{3}$ for any $n \in \mathbb{N}$.

(g) $\sum_{k=1}^{n} k \cdot k! = (n+1)! - 1$ for any $n \in \mathbb{N}$.

(h) $\sum_{k=1}^{n} k \cdot 2^k = (n-1)2^{n+1} + 2$ for any $n \in \mathbb{N}$.

(i) $\left(1 - \frac{1}{2^2}\right) \cdot \left(1 - \frac{1}{3^2}\right) \cdots \left(1 - \frac{1}{n^2}\right) = \frac{n+1}{2n}$ for any $n \in \mathbb{N}$ bigger than 1.

(j) Let $a_1 = 1$ and $a_{n+1} = \sqrt{1 + 2a_n}$ for $n \in \mathbb{N}$. Show that $a_n < 4$ for any $n \in \mathbb{N}$.

(k) For any $n \in \mathbb{N}$ there holds $n \leq 2^n$.

(l) For any $n \in \mathbb{N}$, bigger than 3, there holds $n^2 \leq 2^n$.

(m) For any $n \in \mathbb{N}$, bigger than 6, the inequality $3^n < n!$ holds.

(n) For any $n \in \mathbb{N}$ and real $x \geq -1$ there holds $(1 + x)^n \geq 1 + nx$. [Bernoulli's inequality]

(o) For any $n \in \mathbb{N}$ and all $a, b \in \mathbb{R}$ there holds $(a + b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$. [Binomial theorem]

Results - II. Mathematical proofs

Exercise 1.

(a) Try to draw a suitable right-angled triangle.

(b) There holds $(a \pm b)^2 \geq 0$.

(c) Try to multiply by q.

(d) Add the first and the last term, then second and the second from the last, etc.

(e) Try so-called completing the square.

(f) There holds $a^2 - b^2 = (a - b)(a + b)$ and $a = \sqrt{x}$, $b = \sqrt{y}$.

(g) Show it first for $x \geq 0$ using $a^3 - b^3 = \ldots$ For general case use the fact that $\sqrt[3]{x}$ is odd.

(h) Consider several cases and use the definition of $|x|$.

(i) See Wiki for a lot of proofs.