I. Mathematical logic

Exercise 1. Let A, B be statements. Prove by a truth table that the following are tautologies:

(a) $\neg(\neg A) \Leftrightarrow A$. (Double negation)

(b) $\neg(A \wedge \neg A)$. (Contradiction)

(c) $\neg(A \Rightarrow B) \Leftrightarrow (A \wedge \neg B)$. (Negation of implication)

(d) $(A \Rightarrow B) \Leftrightarrow (\neg B \Rightarrow \neg A)$. (Law of contrapositive)

(e) $A \vee (\neg A)$. (Law of excluded middle)

(f) $(A \Rightarrow B) \Leftrightarrow (\neg A \vee B)$. (Alternative of implication)

(g) $((A \Rightarrow B) \wedge \neg B) \Rightarrow \neg A$. (Modus tollens)

Exercise 2. Let S be the set of all Students and C be the set of all Classes. We consider the following predicates: $L(s,c)$: "Student s Likes c", $H(s,c)$: "Student s Hates c", $G(s,c)$: "Student s is Good at c". Write the following statements using quantifiers, logical connectives and just defined objects.

(a) Every student likes some class.

(b) There exists a class which is liked by all students. [Is it the same as (a)?]

(c) At least one student hates all classes.

(d) If a student hates some class, then (s)he does not like it.

(e) Every student is good at some class and does not hate some other one.

(f) There is a student who is not good at the class which (s)he likes.

(g) If a student is good at some class, then (s)he likes it or does not hate it.

(h) Every student hates at most one class.

(i) There are students who like at least two same classes.

(j) There is a student who likes all classes and is good at three of them.

(k) There is only one student who is good at the class which (s)he does not like.

Exercise 3. Negate the following statements a decide which is true or false:

(a) I do not like this list.

(b) All velociraptors can open windows.

(c) He likes studying or nightlife.

(d) Some mammals lay eggs.

(e) All of us had fish and chips.

(f) If x is even prime, then it is a nice number.

(g) Each woman is loved by some man.

(h) If a student loves cats, than (s)he loves dogs.

(i) $\exists x \in \mathbb{N} : 2 < x < 3$.

(j) $\forall x, y \in \mathbb{R} : x \cdot y \geq 0 \Rightarrow x + y \geq 0$.

(k) $\forall x \in \mathbb{R}\, \exists y \in \mathbb{R} : x \cdot y = 1$.

(l) $\forall x \in \mathbb{R} \setminus \{0\}\, \exists y \in \mathbb{R} : x \cdot y = 1$.

(m) $\forall x \in \mathbb{R}\, \exists y \in \mathbb{Q} : y \leq x \wedge x < y + 1$.

(n) $\forall x \in \mathbb{R}\, \exists y \in \mathbb{N} : y \leq x \wedge x < y + 1$.

(o) $\forall x \in \mathbb{N}\, \exists y \in \mathbb{N}\, \forall z \in \mathbb{N} : z > x \Rightarrow z > y$.

(p) $\exists a, b \in \mathbb{N}\, \forall x \in \mathbb{R} : x > a \Rightarrow -x^2 + bx \leq 0$.

Exercise 4. You are a visitor on an island inhabited only by knights and knaves, where knights always tell the truth and knaves always lie. One day you meet residents A, B and C and ask A what type he is.

(a) A answers: "I am a knave, but B is not." What are A and B?

(b) A answers: "Either I am a knave or B is a knight." What are A and B?

(c) A answers: "B and C are the same type." You ask C: "Are A and B the same type?". What does C answer?

(d) You do not hear A's answer. B then says: "A said that he is the only knight among us" and C says: "Don't believe B, he is lying!". What are B and C?

Results - I. Mathematical logic

Exercise 2.

(a) $\forall s \in S\, \exists c \in C : L(s,c)$.

(b) $\exists c \in C\, \forall s \in S : L(s,c)$. [No.]

(c) $\exists s \in S\, \forall c \in C : H(s,c)$.

(d) $\forall s \in S\, \forall c \in C : H(s,c) \Rightarrow \neg L(s,c)$.

(e) $\forall s \in S\, \exists c_1, c_2 \in C : c_1 \neq c_2 \wedge G(s,c_1) \wedge \neg H(s,c_2)$.

(f) $\exists s \in S\, \exists c \in C : \neg G(s,c) \wedge L(s,c)$.

(g) $\forall s \in S\, \forall c \in C : G(s,c) \Rightarrow (L(s,c) \vee \neg H(s,c))$.

(h) $\forall s \in S\, \forall c_1, c_2 \in C : (H(s,c_1) \wedge H(s,c_2)) \Rightarrow c_1 = c_2$.

(i) $\exists s_1, s_2 \in S\, \exists c_1, c_2 \in C : s_1 \neq s_2 \wedge c_1 \neq c_2 \wedge L(s_1,c_1) \wedge L(s_2,c_1) \wedge L(s_1,c_2) \wedge L(s_2,c_2)$.

(j) $\exists s \in S\, \exists c_1, c_2, c_3 \in C\, \forall c \in C : c_1 \neq c_2 \wedge c_1 \neq c_3 \wedge c_2 \neq c_3 \wedge L(s,c) \wedge G(s,c_1) \wedge \wedge G(s,c_2) \wedge G(s,c_3)$.

(k) $\exists s_1 \in S\, \exists c \in C\, \forall s_2 \in S : (G(s_1,c) \wedge \neg L(s_1,c)) \wedge ((G(s_2,c) \wedge \neg L(s_2,c)) \Rightarrow s_1 = s_2)$.

Exercise 3.

(a) I like this list.

(b) There is a velociraptor, which cannot open windows.

(c) He likes neither studying nor nightlife. (=He does not like studying and he does not like nightlife.)

(d) No mammal lays eggs.

(e) Someone of us had not fish and chips. (There is someone among us, who had not fish or had not chips.)

(f) x is even prime and it is not a nice number.

(g) There is a woman whom any man does not love.

(h) There exists a student, who loves cats, but does not love dogs.

(i) $\forall x \in \mathbb{N} : 2 \geq x \vee x \geq 3$. The negation is true.

(j) $\exists x, y \in \mathbb{R} : x \cdot y \geq 0 \wedge x + y < 0$. The negation is true.

(k) $\exists x \in \mathbb{R}\, \forall y \in \mathbb{R} : x \cdot y \neq 1$. The negation is true.

(l) $\exists x \in \mathbb{R} \setminus \{0\}\, \forall y \in \mathbb{R} : x \cdot y \neq 1$. The original statement is true.

(m) $\exists x \in \mathbb{R}\, \forall y \in \mathbb{Q} : y > x \vee x \geq y + 1$. The original statement is true.

(n) $\exists x \in \mathbb{R}\, \forall y \in \mathbb{N} : y > x \vee x \geq y + 1$. The negation is true.

(o) $\exists x \in \mathbb{N}\, \forall y \in \mathbb{N}\, \exists z \in \mathbb{N} : z > x \wedge z \leq y$. The original statement is true.

(p) $\forall a, b \in \mathbb{N}\, \exists x \in \mathbb{R} : x > a \wedge -x^2 + bx > 0$. The original statement is true.

Exercise 4.

(a) Both A and B are knaves.

(b) Both A and B are knights.

(c) Yes.

(d) B is a knave and C is a knight.