7. Homework

In the following examples, you should investigate the given function f, i.e. you should find:

- The domain and continuity of f.
- Points of intersections with axes.
- Symmetries: oddness, evenness, periodicity.
- Limits at the "endpoints of the domain".
- Asymptotes of the function.
- The intervals of monotonicity; local and global extrema.
- The range of f.
- The intervals of concavity or convexity; inflection points.
- The sketch of the graph of f.

Exercise 1. [4pts] Choose one of the following functions to investigate:

(i) $f(x) = \log^3 x - 3\log^2 x$.

(ii) $f(x) = \frac{1}{x}e^{\frac{2}{x}}$. *Hint:* $\lim_{x\to 0_-} f(x) = 0$ *can be used without the proof.*

(iii) $f(x) = \arctan \frac{x}{1+x^2}$.

(iv) (a) $f(x) = e^{2x} - e^x$ and (b) $f(x) = \frac{1}{1+x^3}$.

If you decide to deal with (iv), you need to investigate both functions.

Bonus 1. [4pts] Choose one of the following functions to investigate:

(i) $f(x) = \sqrt{x^2 - \frac{2}{x}}$.

(ii) $f(x) = \frac{x^3}{x^2+x-2}$. *Hint: It is not necessary to calculate the value of extrema, their sign is enough.*

(iii) $f(x) = \frac{\cos x}{\sin x + 2}$. *Hint:* f *is periodic.*

If you are unsure about something just write me an e-mail. Do not hesitate to check your results with WolframAlpha or GeoGebra.

7. Homework - Solution

Solution to Exercise 1.

(i) Clearly, $\mathcal{D}_f = (0, +\infty)$ and f is continuous on $\mathcal{D}_f$ as a sum (and product) of continuous functions. Due to the shape of its domain, f has no symmetries. There is no intersection with y-axis (because $x \neq 0$). If we set $y = 0$, we are solving $(\log x - 3)\log^2 x = 0$, which gives points $[1, 0]$, $[e^3, 0]$.

Limits. We have

$$\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} (\log x - 3)\log^2 x = +\infty \cdot (+\infty)^2 = +\infty,$$
$$\lim_{x\to 0_+} f(x) = \lim_{x\to 0_+} (\log x - 3)\log^2 x = -\infty \cdot (-\infty)^2 = -\infty,$$
$$\lim_{x\to+\infty} \frac{f(x)}{x} = \lim_{x\to+\infty} \frac{\log^3 x - 3\log^2 x}{x} = 0, \qquad \lim_{x\to+\infty} f(x) - 0\cdot x = +\infty \notin \mathbb{R},$$

and so f has not asymptotes.

1. Derivative. In a standard way, we get

$$f'(x) = 3\frac{\log^2 x}{x} - 6\frac{\log x}{x} = \frac{3\log x}{x}\cdot(\log x - 2),\ x \in \mathcal{D}_f.$$

We thus see that $f' > 0$ for $x \in (0,1)\cup(e^2,+\infty)$ and $f' < 0$ for $x \in (1, e^2)$. It imples that f increases on $(0,1], [e^2, +\infty)$ and decreases on $[1, e^2]$. At $x = 1$ function f has local maximum, $f(1) = 0$ and at $x = e^2$ has local minimum $f(e^2) = -4$. Also, $\mathcal{R}_f = \mathbb{R}$.

2. Derivative. We obtain

$$f''(x) = \left(\frac{3\log^2 x - 6\log x}{x}\right)' = \frac{1}{x^2}\left(\left(\frac{6\log x}{x} - \frac{6}{x}\right)\cdot x - (3\log^2 x - 6\log x)\cdot 1\right)$$
$$= \frac{3}{x^2}\left(2\log x - 2 - \log^2 x + 2\log x\right) = -\frac{3}{x^2}\left(\log^2 x - 4\log x + 2\right),\ x \in \mathcal{D}_f.$$

We need to factor the bracket. We set $z = \log x$ and solve $z^2 - 4z + 2 = 0$, which has roots $z_{1,2} = 2 \pm \sqrt{2}$. Let's denote $x_1 = e^{2-\sqrt{2}}$ and $x_2 = e^{2+\sqrt{2}}$. Now,

$$f''(x) = -\frac{3(\log x - x_1)(\log x - x_2)}{x^2},$$

and therefore, $f'' > 0$ if and only if $x \in (x_1, x_2)$ and $f'' < 0$ for $x \in (0, x_1) \cup (x_2, +\infty)$. It implies that the function f is convex on $(e^{2-\sqrt{2}}, e^{2+\sqrt{2}})$ and concave on $(0, e^{2-\sqrt{2}})$, $(e^{2+\sqrt{2}}, +\infty)$. At both $x = e^{2-\sqrt{2}}$ and $x = e^{2+\sqrt{2}}$ the function f has the inflection points.

We need to realize that $0 < 1 < e^{2-\sqrt{2}} < e^2 < e^{2+\sqrt{2}}$.

(ii) Clearly, $\mathcal{D}_f = \mathbb{R}\setminus\{0\}$ and f is continuous on $\mathcal{D}_f$ as a product (and composition) of continuous functions. Obviously, f has no symmetries. There is no intersection with y-axis (because $x \neq 0$). If we set $y = 0$, we are solving $\frac{1}{x}e^{\frac{2}{x}} = 0$, which has no solution either.

Limits. We have

$$\lim_{x\to\pm\infty} f(x) = \lim_{x\to\pm\infty} \frac{e^{\frac{2}{x}}}{x} = \frac{e^0}{\pm\infty} = 0,$$
$$\lim_{x\to 0_+} f(x) = \lim_{x\to 0_+} \frac{1}{x}e^{\frac{2}{x}} = \frac{+\infty}{0_+} = +\infty, \qquad \lim_{x\to 0_-} f(x) = 0\,\text{(see hint)},$$
$$\lim_{x\to\pm\infty} \frac{f(x)}{x} = \lim_{x\to+\infty} \frac{e^{\frac{2}{x}}}{x^2} = \frac{e^0}{+\infty} = 0, \qquad \lim_{x\to\pm\infty} f(x) - 0\cdot x = 0,$$

and so f has trivial asymptote $y = 0$ at $\pm\infty$.

1. Derivative. In a standard way, we get

$$f'(x) = \frac{1}{x^2}\left(-\frac{2}{x^2}\cdot e^{\frac{2}{x}}\cdot x - e^{\frac{2}{x}}\cdot 1\right) = -\frac{\frac{2}{x}+1}{x^2}\cdot e^{\frac{2}{x}} = -\frac{x+2}{x^3}\cdot e^{\frac{2}{x}},\ x \in \mathcal{D}_f.$$

We thus see that $f' > 0$ for $x \in (-2, 0)$ and $f' < 0$ for $x \in (-\infty, -2) \cup (0, +\infty)$. It imples that f increases on $[-2, 0)$ and decreases on $(-\infty, -2]$, $(0, +\infty)$. At $x = -2$ function f has local (even global) minimum, $f(-2) = -\frac{1}{2e}$. Finally, $\mathcal{R}_f = \left[-\frac{1}{2e}, 0\right) \cup (0, +\infty)$.

2. Derivative. We obtain

$$\begin{aligned} f''(x) &= -\left(\frac{(x+2)e^{\frac{2}{x}}}{x^3}\right)' = -\frac{1}{x^6}\left(\left(e^{\frac{2}{x}} - \frac{2x+4}{x^2}\cdot e^{\frac{2}{x}}\right)\cdot x^3 - (x+2)e^{\frac{2}{x}}\cdot 3x^2\right) \\ &= -\frac{e^{\frac{2}{x}}}{x^4}\left(\frac{x^2-2x-4}{x^2}\cdot x - (3x+6)\right) = -\frac{e^{\frac{2}{x}}}{x^5}\left(x^2-2x-4-3x^2-6x\right) \\ &= \frac{2(x^2+4x+2)}{x^5}\cdot e^{\frac{2}{x}},\ x \in \mathcal{D}_f. \end{aligned}$$

We need to factor the bracket. By quadratic formula we find its roots $x_{1,2} = -2 \pm \sqrt{2}$. Now,

$$f''(x) = \frac{2(x-x_1)(x-x_2)}{x^5}\cdot e^{\frac{2}{x}},$$

and therefore, $f'' > 0$ if and only if $x \in (x_2, x_1) \cup (0, +\infty)$ and $f'' < 0$ for $x \in (-\infty, x_2) \cup (x_1, 0)$. It implies that the function f is convex on $(-2-\sqrt{2}, -2+\sqrt{2})$, $(0, +\infty)$ and concave on $(-\infty, -2-\sqrt{2})$, $(2+\sqrt{2}, 0)$. At both $x_{1,2}$ the function f has the inflection points.

We need to realize that $-2-\sqrt{2} < -2 < -2+\sqrt{2} < 0$.

(iii) Clearly, $\mathcal{D}_f = \mathbb{R}$ and f is continuous on $\mathcal{D}_f$ as a composition of continuous functions. The single intersection point with y-axis is $[0, 0]$ and this is also the only one intersection point with x-axis. Further, we see (because arctan is odd) that $f(-x) = -f(x)$ on the whole domain, i.e. f is odd. It is neither even, nor periodic. We can focus our investigation just on $x \in [0, +\infty)$.

Limits. We have (also, $f(0) = 0$, but we already know it)

$$\begin{aligned} &\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} \arctan\frac{1}{\frac{1}{x}+x} = \arctan\frac{1}{+\infty} = \arctan 0 = 0, \\ &\lim_{x\to+\infty}\frac{f(x)}{x} = \frac{0}{+\infty} = 0, \quad \lim_{x\to+\infty} f(x) - 0\cdot x = 0, \end{aligned}$$

and the function f thus has just trivial asymptote $y = 0$ (at both $\pm\infty$).

1. Derivative. In a standard way, we find

$$f'(x) = \frac{1}{1+\left(\frac{x}{1+x^2}\right)^2}\cdot\frac{1\cdot(1+x^2) - x\cdot 2x}{(1+x^2)^2} = \frac{1-x^2}{(1+x^2)^2+x^2},\ x \in \mathcal{D}_f.$$

We thus see that $f' > 0$ for $x \in (0, 1)$ and $f' < 0$ for $x \in (1, +\infty)$. It imples that f increases on $[0, 1]$ and decreases on $[1, +\infty)$. At $x = 1$ the function f has local maximum $f(1) = \arctan\frac{1}{2}$. Also, $\mathcal{R}_f = \left[-\arctan\frac{1}{2}, \arctan\frac{1}{2}\right]$.

2. Derivative. We directly obtain

$$\begin{aligned} f''(x) &= \frac{-2x\cdot(x^4+3x^2+1) - (1-x^2)\cdot(4x^3+6x)}{((1+x^2)^2+x^2)^2} \\ &= \frac{-2x^5-6x^3-2x+4x^5+6x^3-4x^3-6x}{((1+x^2)^2+x^2)^2} \\ &= \frac{2x^5-4x^3-8x}{((1+x^2)^2+x^2)^2} = \frac{2x(x^4-2x^2-4)}{((1+x^2)^2+x^2)^2},\ x \in \mathcal{D}_f. \end{aligned}$$

We need to factor the bracket in the numerator. We set $z = x^2$ and solve $z^2 - 2z - 4 = 0$, it has roots $z_{1,2} = 1 \pm \sqrt{5}$. Now,

$$f''(x) = \frac{2x(x^2-(1+\sqrt{5}))(x^2-(1-\sqrt{5}))}{((1+x^2)^2+x^2)^2} = \frac{2x(x-x_1)(x+x_2)(x^2+\sqrt{5}-1)}{((1+x^2)^2+x^2)^2},$$

where we set $x_1 = -\sqrt{1+\sqrt{5}}$ and $x_2 = \sqrt{1+\sqrt{5}}$. The second and the third bracket are positive (and the denominator too). Therefore, we have just two zero points of f'' and we see that $f'' > 0$ if and only if $x \in (x_2, +\infty)$ and $f'' < 0$ for $x \in (0, x_2)$. It implies that the function f is convex on $(\sqrt{1+\sqrt{5}}, +\infty)$ and concave on $(0, \sqrt{1+\sqrt{5}})$. At $x = 0$ and x_2 the function f has the inflection points.

We need to realize that $-\sqrt{1+\sqrt{5}} < -1 < 0 < 1 < \sqrt{1+\sqrt{5}}$.

Notes: The common mistake would be forgetting to write that at $x = 0$ f has the inflection point. Do not forget to (correctly) sketch the graph on $\mathbb{R}$, i.e. not just on $[0, +\infty)$.

(iv) (a) Clearly, $\mathcal{D}_f = \mathbb{R}$ and f is continuous on $\mathcal{D}_f$ as a sum of continuous functions. Of course, $f(x) = e^x(e^x - 1)$, and thus, the single intersection point with x-axis is $[0, 0]$ (and it is also the only intersection point with y-axis). The function has no symmetries (it is obvious due to the limits below).

Limits. We have

$$\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} e^x(e^x - 1) = (+\infty)\cdot(+\infty) = +\infty,$$
$$\lim_{x\to-\infty} f(x) = \lim_{x\to+\infty} e^x(e^x - 1) = 0\cdot(-1) = 0$$

and the function f thus has just trivial asymptote $y = 0$ at $-\infty$. At $+\infty$ we have

$$\lim_{x\to+\infty} \frac{f(x)}{x} = \lim_{x\to+\infty} \underbrace{\frac{e^x}{x}}_{\to+\infty,\ \text{Scale}} (e^x - 1) = +\infty \notin \mathbb{R}$$

so there is no asymptote.

1. Derivative. We immediately have

$$f'(x) = 2e^{2x} - e^x = e^x(2e^x - 1)\,,\ x \in \mathcal{D}_f.$$

We thus see that $f' > 0$ for $x \in \left(\log\frac{1}{2}, +\infty\right)$ and $f' < 0$ for $x \in \left(-\infty, \log\frac{1}{2}\right)$. It imples that f increases on $\left[\log\frac{1}{2}, +\infty\right)$ and decreases on $\left(-\infty, \log\frac{1}{2}\right]$. At $x = \log\frac{1}{2}$ the function f has local minimum $f\left(\log\frac{1}{2}\right) = \frac{1}{4} - \frac{1}{2} = -\frac{1}{4}$. Also, $\mathcal{R}_f = \left[\log\frac{1}{2}, +\infty\right]$.

2. Derivative. We obtain

$$f''(x) = 4e^{2x} - e^x = e^x(4e^x - 1)\,,\ x \in \mathcal{D}_f.$$

We quickly see that $f'' > 0$ if and only if $x \in \left(\log\frac{1}{4}, +\infty\right)$ and $f'' < 0$ for $x \in \left(-\infty, \log\frac{1}{4}\right)$. It implies that the function f is convex on $\left(\log\frac{1}{4}, +\infty\right)$ and concave on $\left(-\infty, \log\frac{1}{4}\right)$. At $x = \log\frac{1}{4}$ f has the inflection point.

(b) Clearly, $\mathcal{D}_f = \mathbb{R}\setminus\{-1\}$ and f is continuous on $\mathcal{D}_f$ as a quotient of continuous functions. Obviously, f has no intersection point with x-axis, and $[0, 1]$ is the single intersection point with y-axis. Due to non-symmetric $\mathcal{D}_f$, the function has no symmetries.

Limits. We have

$$\lim_{x\to\pm\infty} f(x) = \frac{1}{1\pm\infty} = 0,$$

so the function f has just the trivial asymptote $y = 0$ at both $\pm\infty$. Around $x = -1$ we find

$$\lim_{x\to-1_+} f(x) = \frac{1}{0_+} = +\infty,$$
$$\lim_{x\to-1_-} f(x) = \frac{1}{0_-} = -\infty.$$

1. Derivative. We immediately have

$$f'(x) = -\frac{3x^2}{(1+x^3)^2}\,,\ x \in \mathcal{D}_f.$$

We thus see that $f' < 0$ on the whole domain. It imples that f decreases on $(-\infty,-1)$, $(-1,+\infty)$. There are no extrema and $\mathcal{R}_f = \mathbb{R}\setminus\{0\}$.

2. Derivative. We obtain

$$f''(x) = -\frac{6x(1+x^3)^2 - 3x^2\cdot 2(1+x^3)\cdot 3x^2}{(1+x^3)^4} = -\frac{6x+6x^4-18x^4}{(1+x^3)^3} = \frac{6x(2x^3-1)}{(1+x^3)^3}\,,\ x\in\mathcal{D}_f.$$

Clearly, f'' has zeros $x=-1,0,\sqrt[3]{\frac{1}{2}}$. Therefore, $f''>0$ for $x\in(-1,0)\cup\left(\sqrt[3]{\frac{1}{2}},+\infty\right)$ and $f''<0$ for $x\in(-\infty,-1)\cup\left(0,\sqrt[3]{\frac{1}{2}}\right)$. It implies that f is convex on $(-1,0)$, $\left(\sqrt[3]{\frac{1}{2}},+\infty\right)$ and concave on $(-\infty,-1)$, $\left(0,\sqrt[3]{\frac{1}{2}}\right)$. At $x=0$, $\sqrt[3]{\frac{1}{2}}$ the function f has the inflection points.

Solution to Exercise 2.

(i) We see that $x^2-\frac{2}{x}\geq 0 \Leftrightarrow \frac{x^3-2}{x}\geq 0$, and therefore, $\mathcal{D}_f=(-\infty,0)\cup[\sqrt[3]{2},+\infty)$ and f is continuous here. There is no intersection with y-axis (as $x\neq 0$) and there is exactly one intersection point with x-axis, the point $[\sqrt[3]{2},0]$. There are no symmetries due to the shape of $\mathcal{D}_f$.

Limits. At infinities we find

$$\lim_{x\to\pm\infty} f(x) = \lim_{x\to\pm\infty}|x|\sqrt{1-\frac{2}{x^3}} = +\infty,\quad \lim_{x\to\pm\infty}\frac{f(x)}{x} = \lim_{x\to\pm\infty}\frac{|x|}{x}\sqrt{1-\frac{2}{x^3}} = \pm 1,$$

$$\lim_{x\to+\infty} f(x)-1\cdot x = \lim_{x\to+\infty}\sqrt{x^2-\frac{2}{x}}-x = \lim_{x\to+\infty}\frac{-\frac{2}{x}}{\sqrt{x^2-\frac{2}{x}}+x} = 0,$$

$$\lim_{x\to-\infty} f(x)-(-1)\cdot x = \lim_{x\to-\infty}\sqrt{x^2-\frac{2}{x}}+x = \lim_{x\to-\infty}\frac{-\frac{2}{x}}{\sqrt{x^2-\frac{2}{x}}-x} = 0$$

and the function f thus has obligue asymptote $y=x$ at $+\infty$, and $y=-x$ at $-\infty$. We also need to find the one-sided limits at points excluded from the domain, i.e.

$$\lim_{x\to 0_-} f(x) = \sqrt{0-\frac{2}{0_-}} = +\infty,\qquad \lim_{x\to\sqrt[3]{2}+} f(x) = f(\sqrt[3]{2}) = 0.$$

1. Derivative. In a straightforward way, we find

$$f'(x) = \frac{2x+\frac{2}{x^2}}{2\sqrt{x^2-\frac{2}{x}}} = \frac{x^3+1}{x^2\sqrt{x^2-\frac{2}{x}}}\quad\text{for } x\in(-\infty,0)\cup(\sqrt[3]{2},+\infty).$$

The denominator is positive (when is defined), so $f'>0$ if and only if $x^3+1>0$ which is for $x>-1$ and $f'<0$ for $x<-1$. Therefore, f is increasing on $[-1,0)$, $[\sqrt[3]{2},+\infty)$ and decreasing on $(-\infty,-1)$. At $x=-1$ and $x=\sqrt[3]{2}$ the function f has local minima, $f(-1)=\sqrt{3}$ and $f(\sqrt[3]{2})=0$. Therefore, $\mathcal{R}_f=[0,+\infty)$.

2. Derivative. In a standard way, we obtain

$$\begin{aligned}
f''(x) &= \frac{1}{x^4\left(x^2-\frac{2}{x}\right)}\left[3x^2\cdot x^2\sqrt{x^2-\frac{2}{x}}-(x^3+1)\cdot\left(2x\sqrt{x^2-\frac{2}{x}}+x^2\frac{x^3+1}{x^2\sqrt{x^2-\frac{2}{x}}}\right)\right]\\
&= \frac{1}{x^4\sqrt{\left(x^2-\frac{2}{x}\right)^3}}\left[3x^4\left(x^2-\frac{2}{x}\right)-(x^3+1)\cdot\left(2x\left(x^2-\frac{2}{x}\right)+x^3+1\right)\right]\\
&= \frac{3x^6-6x^3-(x^3+1)(2x^3-4)-(x^3+1)^2}{x^4\sqrt{\left(x^2-\frac{2}{x}\right)^3}} = \frac{3x^6-6x^3-2x^6+4x^3-2x^3+4-x^6-2x^3-1}{x^4\sqrt{\left(x^2-\frac{2}{x}\right)^3}}\\
&= \frac{3(1-2x^3)}{x^4\sqrt{\left(x^2-\frac{2}{x}\right)^3}},\ x\in\mathcal{D}_f.
\end{aligned}$$

Obviously, $f''>0$ if and only if $1-2x^3>0$, i.e. for $x<\sqrt[3]{\frac{1}{2}}$ and $f''<0$ for $x>\sqrt[3]{\frac{1}{2}}$. Therefore, f is convex on $(-\infty,0)$, and concave on $(\sqrt[3]{2},+\infty)$. There is no inflection point.

(ii) We see that $x^2+x-2=(x+2)(x-1)$, and therefore, $\mathcal{D}_f=\mathbb{R}\setminus\{-2,1\}$ and f is continuous here. If we set $x=0$, we find the intersection point $[0,0]$. Obviously, there are no more of them. There are no symmetries due to the shape of $\mathcal{D}_f$.

Limits. At infinities we find

$$\lim_{x\to\pm\infty} f(x)=\lim_{x\to\pm\infty}\frac{x}{1+\frac{1}{x}-\frac{2}{x^2}}=\pm\infty,\quad \lim_{x\to\pm\infty}\frac{f(x)}{x}=\lim_{x\to\pm\infty}\frac{1}{1+\frac{1}{x}-\frac{2}{x^2}}=1,$$
$$\lim_{x\to\pm\infty} f(x)-1\cdot x=\lim_{x\to\pm\infty}\frac{x}{1+\frac{1}{x}-\frac{2}{x^2}}-x=\lim_{x\to\pm\infty}\frac{-1+\frac{2}{x}}{1+\frac{1}{x}-\frac{2}{x^2}}=-1,$$

and the function f thus has obligue asymptote $y=x-1$ at both $\pm\infty$. We also need to find the one-sided limits at points excluded from the domain, i.e.

$$\lim_{x\to-2_-} f(x)=\lim_{x\to-2_-}\frac{x^3}{(x+2)(x-1)}=\frac{-8}{0_-\cdot(-3)}=-\infty,\quad \lim_{x\to-2_+} f(x)=\frac{-8}{0_+\cdot(-3)}=+\infty,$$
$$\lim_{x\to1_-} f(x)=\frac{1}{3\cdot(0_-)}=-\infty,\quad \lim_{x\to1_+} f(x)=\frac{1}{3\cdot(0_+)}=+\infty.$$

1. Derivative. We directly find

$$f'(x)=\frac{1}{[(x+2)(x-1)]^2}\big(3x^2\cdot(x^2+x-2)-x^3\cdot(2x+1)\big)=\frac{3x^4+3x^3-6x^2-2x^4-x^3}{[(x+2)(x-1)]^2}$$
$$=\frac{x^4+2x^3-6x^2}{[(x+2)(x-1)]^2}=\frac{x^2(x^2+2x-6)}{[(x+2)(x-1)]^2},\ x\in\mathcal{D}_f.$$

We need to factor the polynomial in the numerator. As usual, we find $x_{1,2}=-1\pm\sqrt{7}$. We see that $f'>0$ for $x\in(-\infty,x_2)\cup(x_1,+\infty)$ and $f'<0$ for $x\in(x_2,-2)\cup(-2,1)\cup(1,x_1)$. It imples that f increases on $(-\infty,x_2]$, $[x_1,+\infty)$ and decreases on $[x_2,-2)$, $(-2,1)$, $(1,x_1]$. At $x=-1-\sqrt{7}$ the function f has local maximum $f(x_2)<0$ and At $x=-1+\sqrt{7}$ it has local minimum, $f(x_1)>0$. Also, $\mathcal{R}_f=\mathbb{R}$ (we already know it due to the continuity of f and $\lim_{x\to-2_+} f(x)$, $\lim_{x\to1_-} f(x)$).

We need to realize that $-1-\sqrt{7}<-2<1<-1+\sqrt{7}$.

2. Derivative. In a standard way, we obtain

$$f''(x)=\frac{(4x^3+6x^2-12x)\cdot[(x+2)(x-1)]^2-(x^4+2x^3-6x^2)\cdot 2[(x+2)(x-1)](2x+1)}{[(x+2)(x-1)]^4}$$
$$=\frac{2x(x+2)(x-1)}{[(x+2)(x-1)]^4}\left((2x^2+3x-6)\cdot(\underbrace{x^2+x-2}_{=(x+2)(x-1)})-(x^3+2x^2-6x)\cdot(2x+1)\right)$$
$$=\frac{2x(2x^4+3x^3-6x^2+2x^3+3x^2-6x-4x^2-6x+12-2x^4-4x^3+12x^2-x^3-2x^2+6x)}{[(x+2)(x-1)]^3}$$
$$=\frac{2x(3x^2-6x+12)}{[(x+2)(x-1)]^3}=\frac{6x(x^2-2x+4)}{[(x+2)(x-1)]^3},\ x\in\mathcal{D}_f.$$

As we see, the discriminant of x^2-2x+4 is negative, therefore, $x^2-2x+4>0$ on the whole domain. Signs of f'' are thus determined just by points $-2,0,1$. We see that $f''>0$ for $x\in(-2,0)\cup(1,+\infty)$ and $f''<0$ for $x\in(-\infty,-2)\cup(0,1)$. Therefore, f is convex on $(-2,0),(1,+\infty)$ and concave on $(-\infty,-2),(0,1)$, at $x=0$ the function f has the inflection point.

(iii) We know that $\mathcal{R}_{\sin}=[-1,1]$. It means that the denominator is always defined, i.e. $\mathcal{D}_f=\mathbb{R}$, and f is continuous here. If we set $x=0$, we obtain single intersection point with y-axis, which is $\left[0,\frac{1}{2}\right]$. Next, if $y=0$, we solve $\cos x=0$ which gives us all intersection points with x-axis, these are $\left[\frac{\pi}{2}+k\pi,0\right]$, $k\in\mathbb{Z}$. Finally, both $\cos x$ and $\sin x$ are 2π-periodic, therefore, f is also 2π-periodic function. Also, $f(-x)=\frac{\cos(-x)}{\sin(-x)+2}=\frac{\cos x}{-\sin x+2}\neq\pm f(x)$, which shows that f is neither even, nor odd. Due to the periodicity, we will focus our investigation of f just on the interval $[0,2\pi]$.

Limits. We have $f(0)=f(2\pi)=\frac{1}{2}$ and that is all. Asymptotes cannot exist.

1. Derivative. We get

$$f'(x) = \frac{-\sin x \cdot (\sin x + 2) - \cos x \cdot \cos x}{(\sin x + 2)^2} = \frac{-\sin^2 x - 2\sin x - \cos^2 x}{(\sin x + 2)^2} = -\frac{1 + 2\sin x}{(\sin x + 2)^2}, \; x \in \mathcal{D}_f.$$

We should know that $1 = \sin^2 x + \cos^2 x$, $x \in \mathbb{R}$.

We see that $f' > 0$ if and only if $1 + 2\sin x < 0$, i.e. $\sin x < -\frac{1}{2}$, which happens for $x \in \left(\frac{7\pi}{6}, \frac{11\pi}{6}\right)$. Therefore, f is increasing on $\left[\frac{7\pi}{6}, \frac{11\pi}{6}\right]$ and decreasing on $\left[0, \frac{7\pi}{6}\right]$, $\left[\frac{11\pi}{6}, 2\pi\right]$. At $x = \frac{7\pi}{6}$ the function f has local minimum, $f\left(\frac{7\pi}{6}\right) = \frac{-\frac{\sqrt{3}}{2}}{-\frac{1}{2}+2} = -\frac{1}{\sqrt{3}}$, and at $x = \frac{11\pi}{6}$ f has local maximum, $f\left(\frac{11\pi}{6}\right) = \frac{\frac{\sqrt{3}}{2}}{-\frac{1}{2}+2} = \frac{1}{\sqrt{3}}$. Finally, $\mathcal{R}_f = \left[-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right]$.

2. Derivative. In a standard way, we obtain

$$\begin{aligned} f''(x) &= -\frac{2\cos x \cdot (\sin x + 2)^2 - (1 + 2\sin x) \cdot 2(\sin x + 2)\cos x}{(\sin x + 2)^4} \\ &= -2 \cdot \frac{\cos x \cdot (\sin x + 2) - (1 + 2\sin x) \cdot \cos x}{(\sin x + 2)^3} = -2 \cdot \frac{\sin x \cdot \cos x + 2\cos x - \cos x - 2\sin x \cdot \cos x}{(\sin x + 2)^3} \\ &= \frac{-2(1 - \sin x)\cos x}{(\sin x + 2)^3}, \; x \in \mathcal{D}_f. \end{aligned}$$

We see that $f'' > 0$ exactly when $(1 - \sin x)\cos x < 0$. The bracket is positive up to the situation when it is equal to zero, which happens for $x = \frac{\pi}{2}$ and $\frac{3\pi}{2}$. The second term is negative for $x \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$. So, $f'' > 0$ happens for $x \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$ and $f'' < 0$ is for $x \in \left(0, \frac{\pi}{2}\right) \cup \left(\frac{3\pi}{2}, 2\pi\right)$. Therefore, f is convex on $\left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$ and concave on $\left(0, \frac{\pi}{2}\right)$, $\left(\frac{3\pi}{2}, 2\pi\right)$. At $x = \frac{\pi}{2}$ and $\frac{3\pi}{2}$ the function f has the inflection points.

Finally, everything is extended periodically to the whole $\mathbb{R}$.