6. Homework

Exercise 1. [1pt] Find f' and determine also $\mathcal{D}_f$ and $\mathcal{D}_{f'}$:

$$f(x) = \sqrt[3]{x} \arctan x + \frac{1}{x^4} - 3^x \cos x.$$

Exercise 2. [1pt] Find f' and determine also $\mathcal{D}_f$ and $\mathcal{D}_{f'}$:

$$f(x) = \frac{\log x}{x^2 + e^x}\,.$$

Exercise 3. [1pt] Find f' and determine also $\mathcal{D}_f$ and $\mathcal{D}_{f'}$:

$$f(x) = \arctan^2\left(\log \frac{1}{x^2+1}\right).$$

Exercise 4. [1pt] Find the following limit using L'Hôpital's rule:

$$\lim_{x \to 1} \frac{\sin(\pi x) - 3(x-1)^2}{e^{x^2} - e}\,.$$

Remark. *Be careful about the differentiation of compound functions. Do not forget to write about domains of both f and f'. Finally, you should also check the assumptions of L'Hôpital's rule.*

Bonus 1. [1pt] Find f' and determine also $\mathcal{D}_f$ and $\mathcal{D}_{f'}$:

$$f(x) = \cos(\max\{x^2 + x, 2x + 2\}).$$

Hint: First, find when $x^2 + x \geq 2x + 2$. Then, proceed as for piecewise-defined functions.

Bonus 2. [1pt] Until recently hamburgers at the city sports arena cost $4 each. The food concessionaire sold an average of 10,000 hamburgers on a game night. When the price was raised to $4.40, hamburger sales dropped off to an average of 8000 per night.

(a) Assuming a linear demand curve, find the price of a hamburger that will maximize the nightly hamburger revenue.

(b) If the concessionaire has fixed costs of $1000 per night and the variable cost is $0.60 per hamburger, find the price of a hamburger that will maximize the nightly hamburger profit.

6. Homework - Solution

Solution to Exercise 1. Obviously, $\mathcal{D}_f = \mathbb{R} \setminus \{0\}$ and we have

$$\begin{aligned} f'(x) &= (\sqrt[3]{x})' \arctan x + \sqrt[3]{x}(\arctan x)' + (x^{-4})' - (3^x)' \cos x - 3^x(\cos x)' \\ &= \frac{1}{3\sqrt[3]{x^2}} \arctan x + \frac{\sqrt[3]{x}}{1+x^2} - \frac{4}{x^5} - 3^x \log 3 \cdot \cos x + 3^x \sin x, \end{aligned}$$

which makes sense for all $x \in \mathcal{D}_f$. [1pt]

Solution to Exercise 2. Obviously, $\mathcal{D}_f = (0, +\infty)$ and we have

$$\begin{aligned} f'(x) &= \frac{1}{(x^2+e^x)^2} \left[(\log x)'(x^2+e^x) - \log x \cdot (x^2+e^x)' \right] \\ &= \frac{1}{(x^2+e^x)^2} \left[\frac{x^2+e^x}{x} - (2x+e^x) \log x \right], \end{aligned}$$

which makes sense for all $x \in \mathcal{D}_f$. [1pt]

Solution to Exercise 3. Function arctan is defined everywhere, we thus need $\log \frac{1}{x^2+1}$ to make sense, which is the case for all real numbers because $\frac{1}{x^2+1}$ is positive. Therefore, $\mathcal{D}_f = \mathbb{R}$, and we have

$$\begin{aligned} f'(x) &= 2 \arctan\left(\log \frac{1}{x^2+1}\right) \cdot \left(\arctan\left(\log \frac{1}{x^2+1}\right)\right)' = \frac{2\arctan\left(\log\frac{1}{x^2+1}\right)}{1+\log^2 \frac{1}{x^2+1}} \cdot \left(\log \frac{1}{x^2+1}\right)' \\ &= \frac{2\arctan\left(\log\frac{1}{x^2+1}\right)}{1+\log^2 \frac{1}{x^2+1}} \cdot \frac{1}{\frac{1}{x^2+1}} \cdot \Bigg(\underbrace{\frac{1}{x^2+1}}_{(x^2+1)^{-1}}\Bigg)' = \frac{2\arctan\left(\log\frac{1}{x^2+1}\right)}{1+\log^2 \frac{1}{x^2+1}} \cdot (x^2+1) \cdot \frac{-1}{(x^2+1)^2} \cdot (x^2)' \\ &= -\frac{4x}{x^2+1} \cdot \frac{\arctan\left(\log\frac{1}{x^2+1}\right)}{1+\log^2 \frac{1}{x^2+1}}, \end{aligned}$$

which makes sense for all $x \in \mathcal{D}_f$. [1pt]

Solution to Exercise 4. Due to continuity we see that $\sin(\pi x) - 3(x-1)^2 \to \sin \pi - 0 = 0$ and $e^{x^2} - e \to e - e = 0$ as $x \to 1$. We thus obtain

$$\lim_{x\to 1} \frac{\sin(\pi x) - 3(x-1)^2}{e^{x^2} - e} \overset{\text{L'H}, \frac{0}{0}}{=} \lim_{x \to 1} \frac{\pi \cos(\pi x) - 6(x-1)}{2xe^{x^2}} \overset{\text{cont.}}{=} \frac{\pi \cos \pi - 6 \cdot 0}{2 \cdot 1 \cdot e^1} = -\frac{\pi}{2e}.$$ [1pt]

Solution to Exercise 5. First, let's think about what the function actually looks like. So, what is $\max\{x^2+x, 2x+2\}$ equal to? We have

$$\begin{aligned} x^2 + x &\geq 2x + 2 \\ x^2 - x - 2 &\geq 0 \\ (x-2)(x+1) &\geq 0, \end{aligned}$$

and therefore,

$$f(x) = \begin{cases} \cos(x^2+x), & x \in (-\infty, -1) \cup (2, +\infty) \\ \cos(2x+2), & x \in [-1, 2] \end{cases}.$$

(Note: It does not matter where we put closed brackets.) The function is now in a standard piecewise-defined form and we can differentiate it and obtain

$$f'(x) = \begin{cases} -(2x+1)\sin(x^2+x), & x \in (-\infty, -1) \cup (2, +\infty) \\ -2\sin(2x+2), & x \in (-1, 2) \end{cases}.$$ [0,5pt]

Now, we need to investigate the derivative at $x = -1$ and $x = 2$. We can use either the definition (without any further assumptions) or the theorem about the computation of one-sided derivative. Let's use the later, to apply it we need continuity of f at the respective point. It is kind of obvious, but let's check it via the definition. We have

$$\begin{aligned} f(-1) &= \cos 0 = 1 \\ \lim_{x\to -1_+} f(x) &= \lim_{x\to -1_+} \cos(2x+2) \overset{\text{cont.}}{=} \cos 0 = 1 \\ \lim_{x\to -1_-} f(x) &= \lim_{x\to -1_-} \cos(x^2+x) \overset{\text{cont.}}{=} \cos 0 = 1, \end{aligned}$$

and thus (because limits exist and are equal to the value of $f(-1)$) f is continuous at -1. Similarly,

$$\begin{aligned} f(2) &= \cos(4+2) = \cos 6 \\ \lim_{x\to 2_+} f(x) &= \lim_{x\to 2_+} \cos(x^2+x) \overset{\text{cont.}}{=} \cos 6 \\ \lim_{x\to 2_-} f(x) &= \lim_{x\to 2_-} \cos(2x+2) \overset{\text{cont.}}{=} \cos 6, \end{aligned}$$

and f is continous at 2. Now, we invoke the above-mentioned theorem to compute one-sided derivatives. We have

$$\begin{aligned} f'_+(-1) &= \lim_{x\to -1_+} f'(x) = \lim_{x\to -1_+} -2\sin(2x+2) \overset{\text{cont.}}{=} -2\sin 0 = 0 \\ f'_-(-1) &= \lim_{x\to -1_-} f'(x) = \lim_{x\to -1_-} -(2x+1)\sin(x^2+x) \overset{\text{cont.}}{=} \sin 0 = 0, \end{aligned}$$

and thus (both limits exist and are the same) $f'(-1) = 0$. Similarly,

$$\begin{aligned} f'_+(2) &= \lim_{x\to 2_+} f'(x) = \lim_{x\to 2_+} -(2x+1)\sin(x^2+x) \overset{\text{cont.}}{=} -5\sin 6 \\ f'_-(2) &= \lim_{x\to 2_-} f'(x) = \lim_{x\to 2_-} -2\sin(2x+2) \overset{\text{cont.}}{=} -2\sin 6, \end{aligned}$$

and thus $f'(2)$ does not exist ($\sin 6 \neq 0$ and the values are thus different). [0,5pt]

Solution to Exercise 6. Let us denote by p the price of one hamburger and by q the number of them. Due to assumption of linear dependence, we know that

$$q = ap + b,$$

where a and b are certain (real) parameters. Further, the following needs to be true

$$\begin{aligned} 10000 &= 4a + b, \\ 80000 &= 4.4a + b. \end{aligned}$$

It implies $2000 = -0.4a \Rightarrow a = -5000$, and also $b = 30000$.

(a) Here, the profit is

$$pq = ap^2 + bp = \mathcal{P}(p)$$

and its derivative is simply

$$\mathcal{P}'(p) = 2ap + b = -10000p + 30000 = 0 \Leftrightarrow p = 3.$$

Because $\mathcal{P}$ is a downward-oriented parabola this extremum needs to be maximum and the maximum profit is thus for the price \$3. [0,5pt]

(b) In this case, the profit is

$$pq - 1000 - 0.6q = ap^2 + bp - 1000 - 0.6ap - 0.6b = \mathcal{P}(p).$$

The derivative is

$$\mathcal{P}'(p) = 2ap + b - 0.6a = -10000p + 30000 + 3000 = 0 \Leftrightarrow p = 3.3.$$

For the same reason as before, the maximum profit is thus for the price \$3.30. [0,5pt]