4. Homework

Exercise 1. [2pts] Find the limit:

$$\lim_{x\to 1} \frac{e^{x^3-x}-1}{\arctan(x-1)}\,.$$

Exercise 2. [2pts] Find the limit:

$$\lim_{x\to 0} \left(\sqrt{x+4}-1\right)^{\frac{1}{\sin x}}\,.$$

Remark. *Do not forget to mention the use of arithmetics of limits and continuity; it is enough to indicate it above the relevant equality sign, e.g.* "$\overset{AL}{=}$" *or* "$\overset{cont.}{=}$"*. Further, you have to explain in detail the application of both LCC and LCI.*

Bonus 1. [2pts] Find the limit:

$$\lim_{x\to -\infty} \left(1-\sin\frac{1}{x^2}\right)^{2x^2+x+1}\,.$$

4. Homework - Solution

Solution to Exercise 1. We have

$$\lim_{x\to 1}\frac{e^{x^3-x}-1}{\arctan(x-1)} = \lim_{x\to 1}\underbrace{\frac{e^{x^3-x}-1}{x^3-x}}_{\to 1,\text{ by (1)}}\cdot\underbrace{\frac{1}{\frac{\arctan(x-1)}{x-1}}}_{\to 1,\text{ by (2)}}\cdot\frac{x^3-x}{x-1} \overset{\text{AL}}{=} \lim_{x\to 1}\frac{x(x-1)(x+1)}{x-1}$$

$$= \lim_{x\to 1} x(x+1) \overset{\text{cont.}}{=} 1\cdot 2 = 2. \text{ [1pt]}$$

Ad (1): We use LCI with $g(x) = x^3 - x$ and $f(y) = \frac{e^y-1}{y}$. Of course, $\lim_{x\to 1} g(x) \overset{\text{cont.}}{=} 0$ and $\lim_{y\to 0} f(y) = 1$. The condition (I) asks if $\exists \eta > 0 : g(x) \neq 0\, \forall x \in P(1,\eta)$. Clearly, $g(x) = 0 \Leftrightarrow x \in \{0, \pm 1\}$, and therefore, we take $\eta = \frac{1}{2}$ (or anything < 1). LCI then says that $\lim_{x\to 1} f(g(x)) = 1$. [0,5pt]

Ad (2): We use LCI with $g(x) = x - 1$ and $f(y) = \frac{\arctan y}{y}$. Of course, $\lim_{x\to 1} g(x) \overset{\text{cont.}}{=} 0$ and $\lim_{y\to 0} f(y) = 1$. The condition (I) asks if $\exists \eta > 0 : g(x) \neq 0\, \forall x \in P(1,\eta)$. Clearly, $g(x) = 0 \Leftrightarrow x = 1$, and therefore, we can take any $\eta > 0$. LCI then says that $\lim_{x\to 1} f(g(x)) = 1$. [0,5pt]

Alternative way:

$$\lim_{x\to 1}\frac{e^{x^3-x}-1}{\arctan(x-1)} \overset{\text{L'H},\frac{0}{0}}{=} \lim_{x\to 1}\frac{(3x^2-1)e^{x^3-x}}{1+(x-1)^2} \overset{\text{cont.}}{=} \frac{(3-1)e^0}{1+0} = 2. \text{ [2pt]}$$

Solution to Exercise 2. We have (according to the standard "exponential trick")

$$\lim_{x\to 0}\left(\sqrt{x+4}-1\right)^{\frac{1}{\sin x}} = \lim_{x\to 0} e^{g(x)},$$

where

$$g(x) = \frac{1}{\sin x}\cdot\log\left(\sqrt{x+4}-1\right).$$

We obtain,

$$\lim_{x\to 0} g(x) = \lim_{x\to 0}\underbrace{\frac{1}{\frac{\sin x}{x}}}_{\to 1}\cdot\underbrace{\frac{\log\left(\sqrt{x+4}-1\right)}{\sqrt{x+4}-2}}_{\to 1,\text{ by (1)}}\cdot\frac{\sqrt{x+4}-2}{x} \overset{\text{AL}}{=} \lim_{x\to 0}\frac{\sqrt{x+4}-2}{x}\cdot\frac{\sqrt{x+4}+2}{\sqrt{x+4}+2}$$

$$= \lim_{x\to 0}\frac{x}{x}\cdot\frac{1}{\sqrt{x+4}+2} = \lim_{x\to 0}\frac{1}{\sqrt{x+4}+2} \overset{\text{cont.}}{=} \frac{1}{2+2} = \frac{1}{4}. \text{ [1pt]}$$

Now, let $f(y) = e^y$, then f is continuous at $\frac{1}{4}$, and therefore, by LCC, we get

$$\lim_{x\to 0}\left(\sqrt{x+4}-1\right)^{\frac{1}{\sin x}} = \lim_{x\to 0} f(g(x)) = e^{\frac{1}{4}} = \sqrt[4]{e}. \text{ [0,5pt]}$$

Ad (1): We use LCI with $g(x) = \sqrt{x+4} - 2$ and $f(y) = \frac{\log(y+1)}{y}$. Of course, $\lim_{x\to 0} g(x) \overset{\text{cont.}}{=} 0$ and $\lim_{y\to 0} f(y) = 1$. The condition (I) asks if $\exists \eta > 0 : g(x) \neq 0\, \forall x \in P(0,\eta)$. Clearly ($g$ is monotone), $g(x) = 0 \Leftrightarrow x = 0$, and therefore, we can take any $\eta > 0$. LCI then says that $\lim_{x\to 0} f(g(x)) = 1$. [0,5pt]

Alternative way:

$$\lim_{x\to 0} g(x) \overset{\text{L'H},\frac{0}{0}}{=} \lim_{x\to 0}\frac{\frac{1}{\sqrt{x+4}-1}\cdot\frac{1}{2\sqrt{x+4}}}{\cos x} \overset{\text{cont.}}{=} \frac{\frac{1}{2-1}\cdot\frac{1}{2\cdot 2}}{1} = \frac{1}{4}. \text{ [1,5pt]}$$

Remark. *A lot of you calculated* $\lim_{x\to 0}\log(f(x)) = \frac{1}{4}$. *Then you said that*

$$\lim_{x\to 0}\log(f(x)) = \frac{1}{4}$$
$$\log(\lim_{x\to 0} f(x)) = \frac{1}{4}$$
$$\lim_{x\to 0} f(x) = \sqrt[4]{e}.$$

This is not enough. You need to explain why you can interchange $\lim_{x\to 0}$ *and* $\log$. *This is done using the exponential trick, i.e.*

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} e^{\log f(x)} = \sqrt[4]{e},$$

due to continuity of e^x *at* $\frac{1}{4}$ *and LCC.*

Solution to Exercise 3. We have (according to the standard "exponential trick")

$$\lim_{x\to-\infty}\left(1-\sin\frac{1}{x^2}\right)^{2x^2+x+1}=\lim_{x\to-\infty}e^{g(x)},$$

where

$$g(x)=(2x^2+x+1)\cdot\log\left(1-\sin\frac{1}{x^2}\right).$$

We obtain,

$$\lim_{x\to-\infty}g(x)=\lim_{x\to-\infty}(2x^2+x+1)\cdot\underbrace{\frac{\log\left(1-\sin\frac{1}{x^2}\right)}{-\sin\frac{1}{x^2}}}_{\to 1,\text{ by }(1)}\cdot(-1)\cdot\underbrace{\frac{\sin\frac{1}{x^2}}{\frac{1}{x^2}}}_{\to 1,\text{ by }(2)}\cdot\frac{1}{x^2}$$

$$\overset{\text{AL}}{=}-\lim_{x\to-\infty}\left(2+\frac{1}{x}+\frac{1}{x^2}\right)\overset{\text{AL}}{=}-2.$$

Now, let $f(y)=e^y$, then f is continuous at -2, and therefore, by LCC, we get

$$\lim_{x\to-\infty}\left(1-\sin\frac{1}{x^2}\right)^{2x^2+x+1}=\lim_{x\to-\infty}f(g(x))=e^{-2}=\frac{1}{e^2}.\ \text{[0,5pt]}$$

Ad (1): We use LCI with $g(x)=-\sin\frac{1}{x^2}$ and $f(y)=\frac{\log(y+1)}{y}$. Of course, $\lim_{x\to-\infty}g(x)=0$, see (3) below, and $\lim_{y\to0}f(y)=1$. The condition (I) asks if $\exists\eta>0: g(x)\neq0\,\forall x\in P(-\infty,\eta)$. Clearly, solving

$$-\sin\frac{1}{x^2}=0\text{ for }x\in\left(-\infty,-\frac{1}{\eta}\right)\Leftrightarrow\frac{1}{x}\in(-\eta,0)$$

is equivalent to solving

$$\sin z=0\text{ for }z=\frac{1}{x^2}\in(0,\eta^2).$$

All solutions of this equation are $z=k\pi$, $k\in\mathbb{Z}$. So, it is enough to take $\eta^2=\frac{\pi}{4}$, i.e. $\eta=\frac{\sqrt{\pi}}{2}$. [0,5pt]

Ad (2): We use LCI with $g(x)=\frac{1}{x^2}$ and $f(y)=\frac{\sin y}{y}$. Of course, $\lim_{x\to-\infty}g(x)\overset{\text{AL}}{=}0$ and $\lim_{y\to0}f(y)=1$. The condition (I) asks if $\exists\eta>0: g(x)\neq0\,\forall x\in P(-\infty,\eta)$. Clearly, $g(x)\neq0\,\forall x\in\mathbb{R}$, and therefore, we can take any $\eta>0$. LCI then says that $\lim_{x\to-\infty}f(g(x))=1$. [0,5pt]

Ad (3): We need to verify that $\lim_{x\to-\infty}\sin\frac{1}{x^2}=0$. Let us set $g(x)=\frac{1}{x^2}$ and $f(y)=\sin y$. Of course, $\lim_{x\to-\infty}g(x)\overset{\text{AL}}{=}0$ and f is continous at $y=0$, and therefore, by LCC $\lim_{x\to-\infty}\sin\frac{1}{x^2}=\lim_{x\to-\infty}f(g(x))=0$. [0,5pt]

Alternative way:

$$\lim_{x\to-\infty}g(x)=\lim_{x\to-\infty}\frac{\log\left(1-\sin\frac{1}{x^2}\right)}{\frac{1}{2x^2+x+1}}\underset{\text{see (4)}}{\overset{\text{L'H},\frac{0}{0}}{=}}\lim_{x\to-\infty}\frac{\frac{\frac{2}{x^3}\cos\frac{1}{x^2}}{1-\sin\frac{1}{x^2}}}{-\frac{4x+1}{(2x^2+x+1)^2}}=\lim_{x\to-\infty}-\frac{(2x^2+x+1)^2\frac{2}{x^3}\cos\frac{1}{x^2}}{(4x+1)\left(1-\sin\frac{1}{x^2}\right)}$$

$$=\lim_{x\to-\infty}-2\cdot\underbrace{\frac{\cos\frac{1}{x^2}}{1-\sin\frac{1}{x^2}}}_{\to1,\text{ see (5)}}\cdot\underbrace{\frac{\left(2+\frac{1}{x}+\frac{1}{x^2}\right)^2}{4+\frac{1}{x}}}_{\to\frac{(2+0+0)^2}{4}=1,\text{ by AL}}\overset{\text{AL}}{=}-2\cdot1\cdot1=-2.$$

Ad (4): The fact that $\frac{1}{2x^2+x+1}\to0$ is clear due to AL. We need to check that also $\log\left(1-\sin\frac{1}{x^2}\right)\to0$. We set $g(x)=1-\sin\frac{1}{x^2}$ and $f(y)=\log y$. Due to (5) we know that $\lim_{x\to-\infty}g(x)=1$ and because f is continuous at 1 we use LCC to get $\lim_{x\to-\infty}f(g(x))=\log1=0$.

Ad (5): We need to check that $\cos\frac{1}{x^2}\to1$ and $\sin\frac{1}{x^2}\to0$. See (3) above.

Remark. *In the third exercise, the use of L'Hôpital's rule was not really effective.*