1. Homework

Exercise 1. [1pt] Negate the following statements:

(i) Each continuous function is nice.

(ii) $\exists x \in \mathbb{R} \setminus \{1\} \, \forall n \in \mathbb{N} : x \cdot n > x^2 - x \Rightarrow x \leq 2.$

Exercise 2. [1pt] Use mathematical induction to prove the following statement:

$$\text{For any } n \in \mathbb{N} \text{ there holds } \sum_{k=1}^{n} \frac{1}{k(k+1)} = \frac{n}{n+1}.$$

Exercise 3. [1pt] Find the limit:

$$\lim_{n\to\infty} \frac{3 + n^3 - 4n\sqrt{n}}{2n^3 - n^2 + 2}.$$

Exercise 4. [1pt] Find the limit:

$$\lim_{n\to\infty} n \cdot \left(\sqrt{n^2+2} - \sqrt{n^2-1}\right).$$

Remark. *When calculating limits you need to comment on the use of non-trivial results. So, you do not need to explain why e.g.* $\frac{1}{n^2} \to 0$*, but writing* $\sqrt{1 + \frac{1}{n}^2} \to 1$ *is not enough; you need to explain why it is true (based on some known result). You should also mention the use of arithemitcs of limits (=AL).*

The bonus part follows; you can earn one bonus point for each exercise and also feed your soul.

Bonus 1. [1pt] Find the limit:

$$\lim_{n\to\infty} \frac{\sqrt{n^2 + \sqrt[6]{n} + 2} - \sqrt{n^2 - 2}}{\sqrt[3]{n^2 + \sqrt{n}} - \sqrt[3]{n^2 - \sqrt{n}}}.$$

Bonus 2 (Difficult)**.** [1pt] Let us consider function $f(x) = x^2$ on an interval $I = [0, 2]$. The task is to calculate the area between the curves $y = x^2$ and $y = 0$ for $x \in I$.

(i) Let us fix $N \in \mathbb{N}$ and introduce the finite sequence $\{x_n\}_{n=0}^{N}$ in the form $x_n = \frac{2n}{N}$, i.e. so-called partition of an interval I. We denote $I_n = [x_n, x_{n+1}]$, $n = 0, \dots, N-1$ (i.e. n-th subinterval).

(ii) Draw the picture for $N = 4, 6$.

(iii) Let us now introduce an upper and lower approximation of the area, i.e.

$$U_N = \sum_{n=0}^{N-1} (x_{n+1} - x_n) \cdot \sup_{x \in I_n} f(x) \quad \text{and} \quad L_N = \sum_{n=0}^{N-1} (x_{n+1} - x_n) \cdot \inf_{x \in I_n} f(x).$$

(iv) Realize what are $\sup_{x \in I_n} f(x)$ and $\inf_{x \in I_n} f(x)$.

(v) Explain what U_N and L_N represent for $N = 4, 6$ (look for rectangles).

(vi) What happens for reeeally BIG N?

(vii) Simplify the sums U_N and L_N. You should use the well-known formula for $\sum_{k=1}^{n} k^2$.

(viii) Show that

$$\inf_{N \in \mathbb{N}} U_N = \sup_{N \in \mathbb{N}} L_N = \frac{8}{3}.$$

Based on your pictures you should now believe that this number corresponds to the looked-for quantity. In this way, you just calculated (using the definition) the so-called Riemann integral of f on I. In symbols, $(\mathcal{R}) \int_0^2 f(x) \mathrm{d}x = \frac{8}{3}$. You will get familiar with it in the next semester.

Let me note that there will be a much more effective way of computing such a thing. It will rely on derivatives, which we will study in several weeks.

1. Homework - Solution

Solution to Exercise 1.

(i) There is a continuous function which is not nice./Some continuous function is not nice. [0,5pt]

(ii) $\forall x \in \mathbb{R} \setminus \{1\}\, \exists n \in \mathbb{N} : x \cdot n > x^2 - x \wedge x > 2.$ [0,5pt]

Solution to Exercise 2. As usual, there will be three steps.

(1) First, we check the statement for the first admissible n, in our case it is $n = 1$. So,

$$LHS = \sum_{k=1}^{1} \frac{1}{k(k+1)} = \frac{1}{1(1+1)} = \frac{1}{2} \quad \text{and} \quad RHS = \frac{1}{2}\,.$$

As we see, both quantities are the same and the first step is thus complete.

(2) Now, we suppose that the statement holds for fixed $n \in \mathbb{N}$, i.e. we suppose that

$$\sum_{k=1}^{n} \frac{1}{k(k+1)} = \frac{n}{n+1}\,. \text{ [0,5pt]} \tag{*}$$

(3) Finally, we need to show that the statement holds for $n+1$, it means to verify

$$\sum_{k=1}^{n+1} \frac{1}{k(k+1)} = \frac{n+1}{n+2}\,. \tag{**}$$

We calculate

$$\sum_{k=1}^{n+1} \frac{1}{k(k+1)} = \underbrace{\sum_{k=1}^{n} \frac{1}{k(k+1)}}_{=\frac{n}{n+1} \text{ by } (*)} + \frac{1}{(n+1)(n+2)} = \frac{n}{n+1} + \frac{1}{(n+1)(n+2)}$$
$$= \frac{n(n+2)+1}{(n+1)(n+2)} = \frac{n^2+2n+1}{(n+1)(n+2)} = \frac{(n+1)^2}{(n+1)(n+2)} = \frac{n+1}{n+2}\,.$$

This is exactly the right-hand side of (**) and we are done. [0,5pt]

Solution to Exercise 3. We have

$$\lim_{n\to\infty} \frac{3+n^3-4n\sqrt{n}}{2n^3-n^2+2} = \lim_{n\to\infty} \frac{n^3}{n^3} \cdot \frac{\frac{3}{n^3}+1-\frac{4n^{3/2}}{n^3}}{2-\frac{n^2}{n^3}+\frac{2}{n^3}} = \lim_{n\to\infty} \frac{\frac{3}{n^3}+1-\frac{4}{n^{3/2}}}{2-\frac{1}{n}+\frac{2}{n^3}} \overset{AL}{=} \frac{0+1-0}{2-0+0} = \frac{1}{2}\,. \text{ [1pt]}$$

Solution to Exercise 4. We compute

$$\lim_{n\to\infty} n \cdot (\sqrt{n^2+2} - \sqrt{n^2-1}) = \lim_{n\to\infty} n \cdot (\sqrt{n^2+2} - \sqrt{n^2-1}) \cdot \frac{\sqrt{n^2+2}+\sqrt{n^2-1}}{\sqrt{n^2+2}+\sqrt{n^2-1}}$$
$$= \lim_{n\to\infty} n \cdot \frac{(n^2+2)-(n^2-1)}{\sqrt{n^2}\left(\sqrt{1+\frac{2}{n^2}}+\sqrt{1-\frac{1}{n^2}}\right)} = \lim_{n\to\infty} \frac{n}{n} \cdot \frac{3}{\sqrt{1+\frac{2}{n^2}}+\sqrt{1-\frac{1}{n^2}}}$$
$$= \lim_{n\to\infty} \frac{3}{\sqrt{1+\frac{2}{n^2}}+\sqrt{1-\frac{1}{n^2}}} \overset{AL}{=} \frac{3}{1+1} = \frac{3}{2}\,. \text{ [0,5pt]}$$

We used AL to get

$$1+\frac{2}{n^2} \to 1 \text{ and } 1-\frac{1}{n^2} \to 1 \text{ as } n \to +\infty$$

and thanks to RL we see that

$$\sqrt{1+\frac{2}{n^2}} \to 1 \text{ and } \sqrt{1-\frac{1}{n^2}} \to 1 \text{ as } n \to +\infty. \text{ [0,5pt]}$$

Solution to Exercise 5. For the difference of square roots we get

$$\sqrt{n^2+\sqrt[6]{n}+2}-\sqrt{n^2-2} = \frac{(\sqrt{n^2+\sqrt[6]{n}+2}-\sqrt{n^2-2})(\sqrt{n^2+\sqrt[6]{n}+2}+\sqrt{n^2-2})}{\sqrt{n^2+\sqrt[6]{n}+2}+\sqrt{n^2-2}}$$
$$= \frac{1}{\sqrt{n^2}}\cdot\frac{(n^2+\sqrt[6]{n}+2)-(n^2-2)}{\sqrt{1+\frac{n^{1/6}}{n^2}+\frac{2}{n^2}}+\sqrt{1-\frac{2}{n^2}}} = \frac{\sqrt[6]{n}+4}{n}\cdot\frac{1}{\sqrt{1+\frac{1}{n^{11/6}}+\frac{2}{n^2}}+\sqrt{1-\frac{2}{n^2}}}$$

and for cubic roots we obtain

$$\frac{1}{\sqrt[3]{n^2+\sqrt{n}}-\sqrt[3]{n^2-\sqrt{n}}} = \frac{1}{\sqrt[3]{n^2+\sqrt{n}}-\sqrt[3]{n^2-\sqrt{n}}}\cdot\frac{\sqrt[3]{(n^2+\sqrt{n})^2}+\sqrt[3]{(n^2+\sqrt{n})(n^2-\sqrt{n})}+\sqrt[3]{(n^2-\sqrt{n})^2}}{\sqrt[3]{(n^2+\sqrt{n})^2}+\sqrt[3]{(n^2+\sqrt{n})(n^2-\sqrt{n})}+\sqrt[3]{(n^2-\sqrt{n})^2}}$$
$$= \frac{\sqrt[3]{n^4}}{(n^2+\sqrt{n})-(n^2-\sqrt{n})}\cdot\left[\sqrt[3]{\left((1+\frac{\sqrt{n}}{n^2}\right)^2}+\sqrt[3]{\left(1+\frac{\sqrt{n}}{n^2}\right)\left(1-\frac{\sqrt{n}}{n^2}\right)}+\sqrt[3]{\left(1-\frac{\sqrt{n}}{n^2}\right)^2}\right]$$
$$= \frac{n\cdot\sqrt[3]{n}}{2\sqrt{n}}\cdot\left[\sqrt[3]{\left(1+\frac{1}{n^{3/2}}\right)^2}+\sqrt[3]{\left(1+\frac{1}{n^{3/2}}\right)\left(1-\frac{1}{n^{3/2}}\right)}+\sqrt[3]{\left(1-\frac{1}{n^{3/2}}\right)^2}\right].$$

Together,

$$\lim_{n\to\infty}\frac{\sqrt{n^2+\sqrt[6]{n}+2}-\sqrt{n^2-2}}{\sqrt[3]{n^2+\sqrt{n}}-\sqrt[3]{n^2-\sqrt{n}}} = \lim_{n\to\infty}\frac{\sqrt[6]{n}+4}{n}\cdot\frac{n\cdot\sqrt[3]{n}}{2\sqrt{n}}\cdot\underbrace{\frac{\sqrt[3]{(1+\frac{1}{n^{3/2}})^2}+\sqrt[3]{(1+\frac{1}{n^{3/2}})(1-\frac{1}{n^{3/2}})}+\sqrt[3]{(1-\frac{1}{n^{3/2}})^2}}{\sqrt{1+\frac{1}{n^{11/6}}+\frac{2}{n^2}}+\sqrt{1-\frac{2}{n^2}}}}_{\to\frac{1+1+1}{1+1}=\frac{3}{2},\text{ as } n\to+\infty,\text{ by AL.}}$$
$$\overset{AL}{=}\frac{3}{2}\lim_{n\to\infty}\frac{n^{1/3}(n^{1/6}+4)}{2n^{1/2}} = \frac{3}{2}\lim_{n\to\infty}\frac{n^{3/6}+4n^{2/6}}{2n^{3/6}} = \frac{3}{2}\lim_{n\to\infty}\frac{1+\frac{4}{n^{1/3}}}{2}\overset{AL}{=}\frac{3}{4}.$$ [0,5pt]

In the calculation we used that due to AL we have

$$\left(1+\frac{1}{n^{3/2}}\right)\left(1-\frac{1}{n^{3/2}}\right)\to 1\cdot 1 = 1 \text{ as } n\to+\infty$$

and together with RL it implies

$$\sqrt[3]{\left(1+\frac{1}{n^{3/2}}\right)\left(1-\frac{1}{n^{3/2}}\right)}\to 1 \text{ as } n\to+\infty.$$

Convergence of other roots is analogous. [0,5pt]

Solution to Exercise 6. See the picture below for the sketch of the whole situation for $N=6$. Because f is increasing function on $[0,+\infty)$ we see that (recall that $I_n=[x_n,x_{n+1}]$)

$$\sup_{x\in I_n} f(x) = f(x_{n+1}) \quad\text{and}\quad \inf_{x\in I_n} f(x) = f(x_n).$$

Therefore,

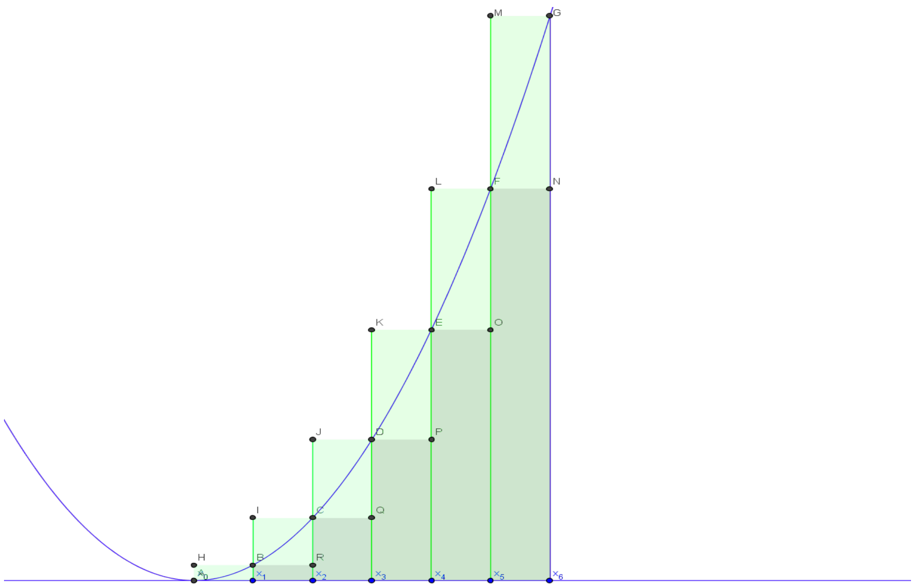
$$(x_{n+1}-x_n)\cdot\sup_{x\in I_n} f(x) = (x_{n+1}-x_n)f(x_{n+1})$$

represents the area of a bigger (green) rectangle. So, e.g. the actual area of x_5x_6GF is over-estimated by it because we add also the area of FGM. Similarly,

$$(x_{n+1}-x_n)\cdot\inf_{x\in I_n} f(x) = (x_{n+1}-x_n)f(x_n)$$

represents smaller (blackish) rectangle. Here, it under-estimate the actual area by ignoring the area of FNG. Hence, U_N represents sum of all bigger rectangles, it is over-estimation of the actual area and L_N is sum of all smaller rectangles and uder-estimation.

Next, if we would take e.g. $N=12$, then the added or neglected areas would be smaller. Therefore, both U_N and L_N would be more precise estimates of the actual area. And so on for bigger N's. [0,5pt]



Observe that we want to take $\lim_{N\to+\infty} U_N$ and $\lim_{N\to+\infty} L_N$. These numbers should be the same and it will be our area. Further, from the construction, U_N needs to be decreasing quantity (in N) and L_N needs to be increasing. Therefore, $\lim_{N\to+\infty} U_N = \inf\limits_{n\in\mathbb{N}} U_N$ and $\lim_{N\to+\infty} L_N = \sup\limits_{n\in\mathbb{N}} L_N$.

Finally, we can evaluate the quantities U_N and L_N, we get

$$\begin{aligned} U_N &= \sum_{n=0}^{N-1} \left(\frac{2(n+1)}{N} - \frac{2n}{N} \right) \cdot x_{n+1}^2 = \frac{2}{N} \sum_{n=0}^{N-1} \frac{4(n+1)^2}{N^2} = \frac{8}{N^3} \sum_{m=1}^{N} m^2 \\ &= \frac{8}{N^3} \cdot \frac{N}{6}(N+1)(2N+1) = \frac{4}{3} \cdot \frac{2N^2+3N+1}{N^2} = \frac{8}{3} \cdot \left(1 + \frac{3}{2N} + \frac{1}{2N^2}\right), \\ L_N &= \sum_{n=0}^{N-1} \left(\frac{2(n+1)}{N} - \frac{2n}{N} \right) \cdot x_n^2 = \frac{2}{N} \sum_{n=0}^{N-1} \frac{4n^2}{N^2} = \frac{8}{N^3} \sum_{n=1}^{N-1} n^2 = \frac{8}{N^3} \cdot \frac{N-1}{6} N(2N-1) \\ &= \frac{4}{3} \cdot \frac{(N-1)(2N-1)}{N^2} = \frac{4}{3} \cdot \frac{2N^2-3N+1}{N^2} = \frac{8}{3} \cdot \left(1 - \frac{3}{2N} + \frac{1}{2N^2}\right). \end{aligned}$$

Now, we can either use limits to get the result (see the previous observation about limits and supremum/infimum) or we make a simple observation. Concerning U_N it is quite obvious (because the bracket is positive) that the smallest possible value is for $N \to +\infty$. Therefore, $\inf_{N\in\mathbb{N}} U_N = \frac{8}{3}$. For L_N it is not so obvious, but it is again true thanks to the following calculation

$$\begin{aligned} -\frac{3}{2N} + \frac{1}{2N^2} &< 0 \\ \frac{1}{N} &< 3 \\ \frac{1}{3} &< N, \end{aligned}$$

which is clearly true for all $N \in \mathbb{N}$. Hence, $\sup_{N\in\mathbb{N}} \left(1 - \frac{3}{2N} + \frac{1}{2N^2}\right) = 1$ and we indeed have

$$\inf_{N\in\mathbb{N}} U_N = \frac{8}{3} \quad \text{and} \quad \sup_{N\in\mathbb{N}} L_N = \frac{8}{3},$$

which was our goal. [0,5pt]