V. Derivace funkcí

Shrnutí teorie.

Definice. (Derivace funkce) Buď f reálná funkce a $a \in \mathbb{R}$. Jestliže existuje

$$\lim_{h\to 0} \frac{f(a+h)-f(a)}{h} ,$$

pak tuto limitu nazýváme **derivací funkce f v bodě** a a značíme ji $f'(a)$. Pomocí příslušné jednostranné limity zavádíme **jednostranné derivace v bodě** a a značíme je $f'_+(a)$ a $f'_-(a)$.

Tvrzení. (Aritmetika derivací) Nechť reálné funkce f a g mají vlastní derivaci v bodě $a \in \mathbb{R}$. Platí:

(i) $(f+g)'(a) = f'(a) + g'(a)$.

(ii) $(f \cdot g)'(a) = f'(a) \cdot g(a) + f(a) \cdot g'(a)$.

(iii) $\left(\frac{f}{g}\right)'(a) = \frac{f'(a)\cdot g(a) - f(a)\cdot g'(a)}{g^2(a)}$, jestliže $g(a) \neq 0$.

Tvrzení. (Derivace složené funkce) Nechť reálná funkce f, resp. g má vlastní derivaci v bodě $y_0 \in \mathbb{R}$, resp. $x_0 \in \mathbb{R}$ a $y_0 = g(x_0)$. Potom platí

$$(f \circ g)'(x_0) = f'(y_0) \cdot g'(x_0).$$

Tvrzení. (Věta o limitě derivací) Nechť reálná funkce f je spojitá zprava v bodě $a \in \mathbb{R}$ a existuje $\lim_{x\to a_+} f'(x)$. Potom existuje $f'_+(a)$ a platí rovnost (analogicky i levostranná varianta)

$$f'_+(a) = \lim_{x\to a_+} f'(x).$$

Příklad 1. (Aritmetika derivací) Najděte derivaci funkce f, určete definiční obory f, f':

(a) $f(x) = 3x^5 - 17x^3 + \sqrt{3}x - 8$.

(b) $f(x) = (x^2 - 2x + 3)e^x$.

(c) $f(x) = 2\cos x + x^2 \sin x$.

(d) $f(x) = (x + \sin x)\arctan x$.

(e) $f(x) = x \log x - \frac{1}{x}\log_3 x$.

(f) $f(x) = \left(\frac{3}{2}\right)^x - x^3\left(\frac{1}{3}\right)^x$.

(g) $f(x) = x^{\frac{3}{4}} + 2\sqrt{x} + \sqrt[4]{x}$.

(h) $f(x) = \frac{\sin x}{x^2-3} + \frac{5x}{x^4+x^2+1}$.

(i) $f(x) = \begin{cases} x, & x \in (-\infty, 0\rangle \\ \log(1+x), & x \in (0,\infty) \end{cases}$.

(j) $f(x) = \begin{cases} 1-x, & x \in (-\infty, 1) \\ (1-x)(2-x), & x \in \langle 1, 2\rangle \\ -(2-x), & x \in (2,\infty) \end{cases}$.

Příklad 2. (Složené funkce) Najděte derivaci funkce f, určete definiční obory f, f':

(a) $f(x) = (x^2 + 51x + 119)^{87}$.

(b) $f(x) = x^3(x+2)^8(x-7)^{11}$.

(c) $f(x) = \log(x^2 + x + 2)$.

(d) $f(x) = \cos(x^3 - x + 2)^9$.

(e) $f(x) = \sin^2 x - \sin(x^2)$.

(f) $f(x) = e^{-\frac{1}{x^2}}$.

(g) $f(x) = \log(\arctan x)$.

(h) $f(x) = \sin(\sin(\sin x))$.

(i) $f(x) = \log(\log^2(\log^3 x))$.

(j) $f(x) = \frac{x^2(1-x)^3}{1+x}$.

(k) $f(x) = \frac{x}{\sqrt{9-x^2}}$.

(l) $f(x) = -\frac{\log x}{\sqrt{x^2-1}}$.

(m) $f(x) = 2\log\frac{x^2-1}{x^2+1}$.

(n) $f(x) = e^{x^2-1} - \log\sqrt{\frac{e^{2x}}{e^{2x}+1}}$.

(o) $f(x) = \left(\frac{1}{x}\right)^{\frac{1}{x}}$.

(p) $f(x) = (\sin x)^{\cos x}$.

(q) $f(x) = \log(\arccos x)$.

(r) $f(x) = \frac{1}{x}\arcsin\sqrt{\frac{x}{x+1}}$.

(s) $f(x) = \arctan\sqrt{x^2-1}$.

(t) $f(x) = x(\arcsin(x^3))^2$.

(u) $f(x) = (\arctan x)^{\arcsin x}$.

(v) $f(x) = \arcsin(\cos x)$.

(w) $f(x) = (x+1)\arccos\frac{2x}{x^2+1}$.

Příklad 3. (Zkouškové příklady) Najděte derivaci funkce f, určete definiční obory f, f':

(a) $f(x) = \cos(\max\{x, x^2\})$.

(b) $f(x) = (x-1)^2 \operatorname{sign}(x^2 + x - 2)$.

(c) $f(x) = \sqrt[3]{3^{|x^2-1|} - 3}$.

(d) $f(x) = (\cos x)^{\min\{2, x^3+x^2-2x+2\}}$.

(e) $f(x) = x^2 \cdot \left\lfloor \frac{4}{\pi}\arctan x \right\rfloor$.

(f) $f(x) = \arctan(x-1) \cdot \left|\arctan^2 x - \frac{\pi^2}{16}\right|$.

(g) $f(x) = \arccos(\min\{2x^2, x+1\})$.

(h) $f(x) = \operatorname{sign}(x^3 - 4x) \cdot \sin(x^3 - 2x^2)$.

(i) $f(x) = \sqrt{1 - (x^2-1)^4}$.

(j) $f(x) = (\cos x - 1)\sqrt{||x| - 2|}$.

Výsledky - V. Derivace funkcí

Příklad 1. (Aritmetika derivací)

(a) $f'(x) = 15x^4 - 51x^2 + \sqrt{3}$, $x \in \mathbb{R}$.

(b) $f'(x) = (x^2 + 1)e^x$, $x \in \mathbb{R}$.

(c) $f'(x) = -2\sin x + 2x \sin x + x^2 \cos x$, $x \in \mathbb{R}$.

(d) $f(x) = \arctan x + \frac{x}{1+x^2} + \cos x \arctan x + \frac{\sin x}{1+x^2}$, $x \in \mathbb{R}$.

(e) $f'(x) = \log x + 1 - \frac{1}{x^2 \log 3} + \frac{1}{x^2} \log_3 x$, $x > 0$.

(f) $f'(x) = \left(\frac{3}{2}\right)^x \log \frac{3}{2} - 3x^2 \left(\frac{1}{3}\right)^x - x^3 \left(\frac{3}{2}\right)^x \log \frac{1}{3}, x \in \mathbb{R}$.

(g) $f'(x) = \begin{cases} \frac{3}{x\sqrt[4]{x}} + \frac{1}{\sqrt{x}} + \frac{1}{4\sqrt[4]{x^3}}, & x > 0 \\ +\infty, & x = 0_+ \end{cases}.$

(h) $f'(x) = \frac{(x^2-3)\cos x - 2x \sin x}{(x^2-3)^2} + \frac{5(x^4+x^2+1) - 5x(4x^3+2x)}{(x^4+x^2+1)^2}$, $x \neq \pm\sqrt{3}$.

(i) $f'(x) = \begin{cases} 1, & x \leq 0 \\ \frac{1}{1+x}, & x > 0 \end{cases}.$

(j) $f'(x) = \begin{cases} -1, & x < 1 \\ 2x - 3, & x \in \langle 1, 2 \rangle \\ 1, & x > 2 \end{cases}.$

Příklad 2. (Složené funkce)

(a) $f'(x) = 87(2x + 51)(x^2 + 51x + 119)^{86}, x \in \mathbb{R}$.

(b) $f'(x) = (22x^2 - 49x - 42)x^2(x + 2)^7(x - 7)^{10}, x \in \mathbb{R}$.

(c) $f'(x) = \frac{2x+1}{x^2+x+2}, x \in \mathbb{R}$.

(d) $f'(x) = 9(1 - 3x^2)(x^3 - x + 2)^8 \sin(x^3 - x + 2)^9, x \in \mathbb{R}$.

(e) $f'(x) = 2\sin x \cos x - 2x \cos(x^2), x \in \mathbb{R}$.

(f) $f'(x) = \frac{2}{x^3} e^{-\frac{1}{x^2}}, x \in \mathbb{R} \setminus \{0\}$.

(g) $f'(x) = \frac{1}{\arctan x} \cdot \frac{1}{1+x^2}$, $x > 0$.

(h) $f'(x) = \cos[\sin(\sin x)] \cdot \cos(\sin x) \cdot \cos x, x \in \mathbb{R}$.

(i) $f'(x) = \frac{6}{x \cdot \log x \cdot \log(\log^3 x)}, x \in (1, e) \cup (e, +\infty)$.

(j) $f'(x) = \frac{-2x(2x^2+2x-1)(1-x)^2}{(1+x)^2}, x \in \mathbb{R} \setminus \{-1\}$.

(k) $f'(x) = \frac{9}{(9-x^2)^{\frac{3}{2}}}, x \in (-3, 3)$.

(l) $f'(x) = \frac{x^2 \log x + 1 - x^2}{x(x^2-1)^{\frac{3}{2}}}, x > 1$.

(m) $f'(x) = \begin{cases} \frac{8x}{x^4-1}, & x \in (-\infty, -1) \cup (1, +\infty) \\ -\infty, & x = -1_- \\ +\infty, & x = 1_+ \end{cases}.$

(n) $f'(x) = 2xe^{x^2-1} - \frac{1}{e^{2x}-1}, x \in \mathbb{R}$.

(o) $f'(x) = -\frac{1}{x^2}(1 + \log \frac{1}{x}) \left(\frac{1}{x}\right)^{\frac{1}{x}}, x > 0$.

(p) $f'(x) = (-\sin x \cdot \log(\sin x) + \frac{\cos^2 x}{\sin x})(\sin x)^{\cos x}$, $x \in \bigcup_{k \in \mathbb{Z}} (2k\pi, (2k + 1)\pi)$.

(q) $f'(x) = \begin{cases} -\frac{1}{\arccos x} \cdot \frac{1}{\sqrt{1-x^2}}, & x \in (-1, 1) \\ -\infty, & x = -1_+ \end{cases}.$

(r) $f'(x) = -\frac{1}{x^2}\arcsin\sqrt{\frac{x}{x+1}} + \frac{\sqrt{x}}{2x^2(x+1)}$, $x > 0$.

(s) $f'(x) = \begin{cases} \frac{1}{x\sqrt{x^2-1}}, & |x| > 1 \\ +\infty, & x = 1_+ \\ -\infty, & x = 1_- \end{cases}.$

(t) $f'(x) = \begin{cases} (\arcsin(x^3))^2 + \frac{6x^3\cdot\arcsin(x^3)}{\sqrt{1-x^6}}, & x \in (-1,1) \\ +\infty, & x = -1_+ \\ +\infty, & x = 1_- \end{cases}.$

(u) $f'(x) = \begin{cases} (\arctan x)^{\arcsin x}\left(\frac{\log(\arctan x)}{\sqrt{1-x^2}} + \frac{\arcsin x}{\arctan x}\cdot\frac{1}{1+x^2}\right), & x \in (0,1) \\ -\infty, & x = 1_- \end{cases}.$

(v) $f'(x) = \begin{cases} -\text{sgn }(\sin x), & x \in \mathbb{R}\setminus\{k\pi,\, k\in\mathbb{Z}\} \\ -1, & x \in \{2k\pi_+, (2k+1)\pi_-\},\, k\in\mathbb{Z} \\ 1, & x \in \{2k\pi_-, (2k+1)\pi_+\},\, k\in\mathbb{Z} \end{cases}.$

(w) $f'(x) = \begin{cases} \arccos\frac{2x}{x^2+1} + 2\text{sign}\,(x^2-1)\frac{x+1}{x^2+1}, & x \in \mathbb{R}\setminus\{\pm1\} \\ \pi, & x = -1 \\ 2, & x = 1_+ \\ -2, & x = 1_- \end{cases}.$

Příklad 3. (Zkouškové příklady)

(a) $f'(x) = \begin{cases} -\sin x, & x \in (0,1) \\ -2x\sin(x^2), & x \in (-\infty,0)\cup(1,+\infty) \\ 0, & x = 0 \\ -\sin 1, & x = 1_- \\ -2\sin 1, & x = 1_+ \end{cases}.$

(b) $f'(x) = \begin{cases} 2(x-1)\text{sign}\,(x^2+x-2), & x \in \mathbb{R}\setminus\{-2,1\} \\ 2, & x = 1 \\ 6, & x = -2_+ \\ -6, & x = -2_- \end{cases}.$

(c) $f'(x) = \begin{cases} \frac{\log 3\cdot 3^{|x^2-1|}\cdot 2x\cdot\text{sgn }(x^2-1)}{3(3^{|x^2-1|}-3)^{\frac{2}{3}}}, & x \in \mathbb{R}\setminus\{0,\pm1,\pm\sqrt{2}\} \\ -\infty, & x \in \{-\sqrt{2}, 0_+\} \\ +\infty, & x \in \{\sqrt{2}, 0_-\} \\ -\frac{\sqrt[3]{2}}{3}\log 3, & x \in \{1_-, -1_+\} \\ \frac{\sqrt[3]{2}}{3}\log 3, & x \in \{1_+, -1_-\} \end{cases}.$

(d) $f'(x) = \begin{cases} e^{g(x)\log(\cos x)}\cdot(g'(x)\cdot\log(\sin x) - g(x)\cdot\tan x), & x \in \bigcup_{k\in\mathbb{Z}}(-\frac{\pi}{2}+2k\pi, \frac{\pi}{2}+2k\pi)\setminus\{0,1\} \\ 0, & x = 0 \\ -2\cos^2 1\cdot\tan 1, & x = 1_+ \\ -\cos^2 1\cdot(2\tan 1 - 3\log(\cos 1)), & x = 1_- \end{cases}.$

(e) $f'(x) = \begin{cases} -4x, & x \in (-\infty,-1) \\ -2x, & x \in (-1,0) \\ 0, & x \in (0,1) \\ 2x, & x \in (1,\infty) \\ 0, & x = 0 \\ +\infty, & x \in \{-1_-, 1_-\} \\ 0, & x \in \{-1_+, 1_+\} \end{cases}.$

(f) $f'(x) = \begin{cases} \frac{|\arctan^2 x - \frac{\pi^2}{16}|}{1+(1-x)^2} + \arctan(x-1)\cdot\text{sgn }(\arctan^2 x - \frac{\pi^2}{16})\cdot\frac{2\arctan x}{1+x^2}, & x \in \mathbb{R}\setminus\{\pm1\} \\ 0, & x = 1 \\ -\frac{\pi}{4}\arctan 2, & x = -1_+ \\ \frac{\pi}{4}\arctan 2, & x = -1_- \end{cases}.$

(g) $$f'(x) = \begin{cases} \frac{-1}{\sqrt{1-(x+1)^2}}, & x \in (-2, -\frac{1}{2}) \\ \frac{-4x}{\sqrt{1-4x^4}}, & x \in (-\frac{1}{2}, \frac{\sqrt{2}}{2}) \\ -\infty, & x = -2_+ \\ -\infty, & x = (\frac{\sqrt{2}}{2})_- \\ \frac{4}{\sqrt{3}}, & x = (\frac{-1}{2})_+ \\ -\frac{2}{\sqrt{3}}, & x = (\frac{-1}{2})_- \end{cases}.$$

(h) $$f'(x) = \begin{cases} (3x^2 - 4x) \cdot \operatorname{sgn}\,(x^3 - 4x) \cdot \cos(x^3 - 2x^2), & x \in \mathbb{R} \setminus \{0, \pm 2\} \\ 0, & x = 0 \\ -4, & x = 2_- \\ 4, & x = 2_+ \\ \infty, & x = -2 \end{cases}.$$

(i) $$f'(x) = \begin{cases} -\frac{4x(x^2-1)^3}{\sqrt{1-(x^2-1)^4}}, & x \in (-\sqrt{2}, 0) \cup (0, \sqrt{2}) \\ -2, & x = 0_- \\ 2, & x = 0_+ \\ +\infty, & x = -\sqrt{2}+ \\ -\infty, & x = \sqrt{2}- \end{cases}.$$

(j) $$f'(x) = \begin{cases} \frac{\operatorname{sign}\,(|x|-2)}{2\sqrt{||x|-2|}} \cdot \operatorname{sign} x \cdot (\cos x - 1) - \sqrt{||x| - 2|} \sin x, & x \in \mathbb{R} \setminus \{0, \pm 2\} \\ 0, & x = 0 \\ -\infty, & x = \pm 2_+ \\ +\infty, & x = \pm 2_- \end{cases}.$$