Real functions

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Contents

Part I

Winter semester

Chapter 1

Differentiation of measures

1.1 Covering theorems

Covering theorems provide a tool which enables us to infer global properties from local ones in the context of measure theory.

Vitali theorem

Definition. Let $A \subset \mathbf{R}^n$. We say that a system $\mathcal{V}$ consisting of closed balls from $\mathbf{R}^n$ forms **Vitali cover of** A, if

$$\forall x \in A\, \forall \varepsilon > 0\, \exists B \in \mathcal{V} \colon x \in B \ \wedge \ \operatorname{diam} B < \varepsilon.$$

Notation.

- $\lambda_n \ldots$ Lebesgue measure on $\mathbf{R}^n$
- $\lambda_n^* \ldots$ outer Lebesgue measure on $\mathbf{R}^n$
- If $B \subset \mathbf{R}^n$ is a ball and $\alpha > 0$, then $\alpha \star B$ denotes the ball, which is concentric with B and with α-times greater radius than B.

Theorem 1.1 (Vitali)**.** *Let $A \subset \mathbf{R}^n$ and $\mathcal{V}$ be a system of closed balls forming a Vitali cover of A. Then there exists a countable disjoint subsystem $\mathcal{A} \subset \mathcal{V}$ such that $\lambda_n(A \setminus \bigcup \mathcal{A}) = 0$.*

Proof. First assume that A is bounded. Take an open bounded set $G \subset \mathbf{R}^n$ with $A \subset G$. Set

$$\mathcal{V}^* = \{B \in \mathcal{V};\ B \subset G\}.$$

The system $\mathcal{V}^*$ is a Vitali cover of A again. If there exists a finite disjoint subsystem $\mathcal{V}^*$ covering A, we are done. So assume

($\star$) there is no finite disjoint subsystem of $\mathcal{V}^*$ covering A.

1st step. We set
$$s_1 = \sup\{\operatorname{diam} B; B \in \mathcal{V}^*\}$$
and choose a ball $B_1 \in \mathcal{V}^*$ such that $\operatorname{diam} B_1 > s_1/2$. We know that $\mathcal{V}^* \neq \emptyset$ and $s_1 \leq \operatorname{diam} G < \infty$.

k-th step. Suppose that we have already chosen balls $B_1, \dots, B_{k-1}$. We set
$$s_k = \sup\Big\{\operatorname{diam} B;\ B \in \mathcal{V}^* \ \wedge\ B \cap \bigcup_{i=1}^{k-1} B_i = \emptyset\Big\}.$$

The supremum is considered for a nonempty set since the set $\bigcup_{i=1}^{k-1} B_i$ is closed, which by ($\star$) does not cover A, and $\mathcal{V}^*$ is a Vitali cover of A. We choose a ball $B_k \in \mathcal{V}^*$ such that $B_k \cap \bigcup_{i=1}^{k-1} B_i = \emptyset$ and $\operatorname{diam} B_k > s_k/2$.

This finishes the construction of the sequence $(B_k)_{k=1}^{\infty}$. Set $\mathcal{A} = \{B_k;\ k \in \mathbf{N}\}$. We verify that $\mathcal{A}$ is the desired system.

- *$\mathcal{A}$ is countable.* This follows immediately from the construction.

- *$\mathcal{A}$ is disjoint.* This follows from the construction.

- *It holds $\lambda_n(A \setminus \bigcup \mathcal{A}) = 0$.* We have
$$\sum_{i=1}^{\infty} \lambda_n(B_i) = \lambda_n\Big(\bigcup_{i=1}^{\infty} B_i\Big) \leq \lambda_n(G) < \infty.$$

Thus the series $\sum_{i=1}^{\infty} \lambda_n(B_i)$ is convergent, therefore $\lim_i \lambda_n(B_i) = 0$. Using the fact that B_i, $i \in \mathbf{N}$, are balls we also have $\lim_i \operatorname{diam} B_i = 0$. We know that $2 \operatorname{diam} B_i > s_i$, consequently $\lim_i s_i = 0$.

We show that
$$\forall x \in A \setminus \bigcup \mathcal{A}\ \forall i \in \mathbf{N}\ \exists j \in \mathbf{N}, j > i :\ x \in 5 \star B_j.$$

Take $x \in A \setminus \bigcup \mathcal{A}$ and $i \in \mathbf{N}$. Denote $\delta = \operatorname{dist}(x, \bigcup_{k=1}^{i} B_k)$. It holds $\delta > 0$ and there exists $B \in \mathcal{V}^*$ such that $x \in B$ and $\operatorname{diam} B < \delta$. Then we have $B \cap \bigcup_{k=1}^{i} B_k = \emptyset$. Thus we have $\operatorname{diam} B > s_p$ for some $p \in \mathbf{N}$ since $\lim_i s_i = 0$. Therefore there exists $j > i$ with $B_j \cap B \neq \emptyset$. Let j be the smallest number with this property. Then we have $s_j \geq \operatorname{diam} B$ since $B \cap \bigcup_{l=1}^{j-1} B_l = \emptyset$. Further we have $\operatorname{diam} B_j > s_j/2 \geq \frac{1}{2} \operatorname{diam} B$. Together we have $2 \operatorname{diam} B_j \geq \operatorname{diam} B$. This implies $x \in B \subset 5 \star B_j$.

For any $i \in \mathbf{N}$ we have
$$\lambda_n^*(A \setminus \bigcup \mathcal{A}) \leq \lambda_n\big(\bigcup_{j=i}^{\infty} 5 \star B_j\big) \leq \sum_{j=i}^{\infty} \lambda_n(5 \star B_j) = 5^n \sum_{j=i}^{\infty} \lambda_n(B_j).$$

Using $\lim_{i\to\infty}\sum_{j=i}^{\infty}\lambda_n(B_j)=0$ we get $\lambda_n^*(A\setminus\bigcup\mathcal{A})=0$, and therefore $\lambda_n(A\setminus\bigcup\mathcal{A})=0$.

Now we assume that the set A is a general subset of $\mathbf{R}^n$. Let $(G_j)_{j=1}^{\infty}$ be a sequence of bounded disjoint open sets such that $\lambda_n(\mathbf{R}^n\setminus\bigcup_{j=1}^{\infty}G_j)=0$. Denote

$$\mathcal{V}_j^*=\{B\in\mathcal{V};\ B\subset G_j\}.$$

The system $\mathcal{V}_j^*$ forms a Vitali cover of the bounded set $G_j\cap A$. Using the previous part of the construction we find a countable disjoint system $\mathcal{A}_j\subset\mathcal{V}_j^*$ with $\lambda_n\big((G_j\cap A)\setminus\bigcup\mathcal{A}_j\big)=0$. Now we set $\mathcal{A}=\bigcup_j\mathcal{A}_j$. □

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Definition. We say that a measure μ on $\mathbf{R}^n$ satisfies **Vitali theorem**, if for every $M\subset\mathbf{R}^n$ and every Vitali cover $\mathcal{V}$ of M there exists countable disjoint cover $\mathcal{A}\subset\mathcal{V}$ such that $\mu\big(M\setminus\bigcup\mathcal{A}\big)=0$.

Remark. (1) By Theorem 1.1 λ_n satisfies Vitali theorem.

(2) If μ satisfies Vitali theorem and $\nu\ll\mu$, then ν satisfies Vitali theorem.

Remark. If μ is the Borel measure on $\mathbf{R}^2$ such that $\mu(A)=\lambda_1\big(A\cap(\mathbf{R}\times\{0\})\big)$ for any $B\subset\mathbf{R}^2$ Borel, then Vitali theorem does not hold for μ.

Theorem 1.2. *Let $E\subset\mathbf{R}^n$ be measurable and $\mathcal{S}$ be a finite system of closed balls covering E. Then there exists a disjoint system $\mathcal{L}\subset\mathcal{S}$ such that $\lambda_n(E)\le 3^n\sum_{B\in\mathcal{L}}\lambda_n(B)$.*

Proof. Without any loss of generality we may assume that $\mathcal{S}$ is nonempty. Choose $B_1\in\mathcal{S}$ with maximal radius among balls in $\mathcal{S}$. Suppose that we have already constructed $B_1,\dots,B_{k-1}$. If possible, choose $B_k\in\mathcal{S}$ disjoint with $\bigcup_{i<k}B_i$ and with maximal radius among balls in $\mathcal{S}$ satisfying this property. We construct a finite sequence of closed balls $B_1,\dots,B_N$ and set $\mathcal{L}=\{B_1,\dots,B_N\}$. We have $E\subset\bigcup_{B\in\mathcal{L}}3\star B$. To this end consider $x\in E$. Then there exists $B\in\mathcal{S}$ with $x\in B$. We find minimal k such that $B\cap B_k\neq\emptyset$. Then we have $\operatorname{radius}(B)\le\operatorname{radius}(B_k)$. This implies that $x\in B\subset 3\star B_k$.

Then we have

$$\lambda_n(E)\le\lambda_n\Big(\bigcup_{B\in\mathcal{L}}3\star B\Big)\le\sum_{B\in\mathcal{L}}\lambda_n(3\star B)=3^n\sum_{B\in\mathcal{L}}\lambda_n(B).$$

□

Besicovitch theorem

Theorem 1.3 (Besicovitch)**.** *For each $n\in\mathbf{N}$ there exists $N\in\mathbf{N}$ with the following property. If $A\subset\mathbf{R}^n$ and $\Delta\colon A\to(0,\infty)$ is a bounded function, then there exist sets $A_1,\dots,A_N$ such that*

- $\big\{\overline{B}(x,\Delta(x));\ x\in A_i\big\}$ *is disjoint for every $i\in\{1,\dots,N\}$,*

- $A \subset \bigcup\{\overline{B}(x, \Delta(x));\ x \in \bigcup_{i=1}^{N} A_i\}$.

Proof. The case of a bounded set A. Let $R = \sup_A \Delta$. Choose $B_1 := \overline{B}(a_1, r_1)$ such that $a_1 \in A$ and $r_1 := \Delta(a_1) > \frac{3}{4}R$. Assume that we have already chosen balls $B_1, \dots, B_{j-1}$ where $j \geq 2$. If

$$F_j := A \setminus \bigcup_{i=1}^{j-1} \overline{B}(a_i, r_i) = \emptyset,$$

then the process stops and we set $J = j$. If $F_j \neq \emptyset$, we continue by choosing $B_j := \overline{B}(a_j, r_j)$ such that $a_j \in F_j$ and

$$r_j := \Delta(a_j) > \tfrac{3}{4} \sup_{F_j} \Delta. \tag{1.1}$$

If $F_j \neq \emptyset$ for all j, then we set $J = \infty$. In this case $\lim_{j\to\infty} r_j = 0$ because A is bounded and the inequalities

$$\|a_i - a_j\| \geq r_i = \frac{1}{3}r_i + \frac{2}{3}r_i > \frac{1}{3}r_i + \frac{1}{2}r_j > \frac{1}{3}r_i + \frac{1}{3}r_j$$

for $i < j < J$ imply that

$$\left\{\tfrac{1}{3} \star B_j;\ j < J\right\} \text{ is a disjoint family.} \tag{1.2}$$

In case $J < \infty$, we have $A \subset \bigcup_{j<J} B_j$. This is also true in the case $J = \infty$. Otherwise there exist $a \in \bigcap_{j=1}^{\infty} F_j$ and $j_0 \in \mathbf{N}$ with $r_{j_0} \leq \frac{3}{4}\Delta(a)$, contradicting the choice of r_{j_0}.

Fix $k < J$. We set $I = \{i < k;\ B_i \cap B_k \neq \emptyset\}$. We now prove that there exists $M \in \mathbf{N}$ depending only on n which estimates $|I|$. To this end we split I into I_1 and I_2 and we estimate their cardinality separately.

$$I_1 = \{i < k;\ B_i \cap B_k \neq \emptyset, r_i < 10r_k\},$$
$$I_2 = \{i < k;\ B_i \cap B_k \neq \emptyset, r_i \geq 10r_k\}.$$

The estimate of $|I_1|$. We have $\frac{1}{3} \star B_i \subset 15 \star B_k$ for every $i \in I_1$. Indeed, if $x \in \frac{1}{3} \star B_i$, then

$$\|x - a_k\| \leq \|x - a_i\| + \|a_i - a_k\| \leq \frac{10}{3}r_k + r_i + r_k \leq \frac{43}{3}r_k < 15r_k.$$

Hence, there are at most 60^n elements of I_1, because for any $i \in I_1$ we have

$$\lambda_n(\tfrac{1}{3} \star B_i) = \lambda_n(\overline{B}(0,1)) \cdot \left(\tfrac{1}{3}r_i\right)^n > \lambda_n(\overline{B}(0,1)) \cdot \left(\tfrac{1}{4}r_k\right)^n = \frac{1}{60^n}\lambda_n(15 \star B_k).$$

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See 1.7.

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The estimate of $|I_2|$. Denote $b_i = a_i - a_k$. An elementary mesh-like construction gives a family $\{Q_m;\ 1 \le m \le (22n)^n\}$ of closed cubes with edge length $1/(11n)$ (so that $\operatorname{diam} Q_m \le 1/11$), which cover $[-1,1]^n$ and thus in particular the unit sphere. We claim that for each $1 \le m \le (22n)^n$ there is at most one $i \in I_2$ such that $b_i/\|b_i\| \in Q_m$, which estimates the cardinality of I_2.

If the claim were not valid, then there would exist $i, j \in I_2, i < j$, such that

$$\left\| \frac{b_i}{\|b_i\|} - \frac{b_j}{\|b_j\|} \right\| \le \frac{1}{11}.$$

Notice that

$$r_i < \|b_i\| < r_i + r_k \quad \text{and} \quad r_j < \|b_j\| < r_j + r_k, \tag{1.3}$$

as the balls B_i, B_j intersect B_k but does not contain a_k. Hence

$$\big| \|b_i\| - \|b_j\| \big| \le |r_i - r_j| + r_k \le |r_i - r_j| + \frac{1}{10} r_j.$$

and

$$\|b_j\| \le r_j + r_k \le r_j + \frac{1}{10} r_j = \frac{11}{10} r_j. \tag{1.4}$$

We have

$$\begin{aligned}
\|a_i - a_j\| = \|b_i - b_j\| &\le \left\| b_i - \frac{\|b_j\|}{\|b_i\|} b_i \right\| + \left\| \frac{\|b_j\|}{\|b_i\|} b_i - b_j \right\| \\
&= \left\| \frac{\|b_i\| b_i}{\|b_i\|} - \frac{\|b_j\|}{\|b_i\|} b_i \right\| + \left\| \frac{\|b_j\|}{\|b_i\|} b_i - \frac{\|b_j\|}{\|b_j\|} b_j \right\| \\
&\le \big| \|b_i\| - \|b_j\| \big| + \frac{1}{11} \|b_j\| \\
&\le |r_i - r_j| + \frac{1}{10} r_j + \frac{1}{10} r_j \qquad \text{(using (1.3) and (1.4))} \\
&\le \begin{cases} r_i - \frac{4}{5} r_j < r_i & \text{if } r_i > r_j, \\ -r_i + \frac{6}{5} r_j \le -r_i + \frac{8}{5} r_i < r_i & \text{if } r_i \le r_j. \end{cases}
\end{aligned}$$

In the last inequality we have used that $i < j$ and thus $r_j < \frac{4}{3} r_i$ by (1.1). We arrived at a contradiction as $i < j$ and thus $a_j \notin B_i$. Hence $|I_2| \le (22n)^n$.

Thus it is sufficient to choose $M > 60^n + (22n)^n$.

Choice of $A_1, \dots, A_M$. For each $k \in \mathbf{N}$ we define $\lambda_k \in \{1, 2, \dots, M\}$ such that $\lambda_k = k$ whenever $k \le M$ and for $k > M$ we define λ_k inductively as follows. There is $\lambda_k \in \{1, \dots, M\}$ such that

$$B_k \cap \bigcup \{B_i;\ i < k, \lambda_i = \lambda_k\} = \emptyset.$$

Now we set $A_j = \{a_i;\ \lambda_i = j\}$, $j = 1, \dots, M$.

The case of a general set A. For each $l \in \mathbf{N}$ apply the previously obtained result with A replaced by

$$A^l = A \cap \{x;\ 3(l-1)R \le \|x\| < 3lR\},$$

and denote resulting sets as $A_i^l,\ i = 1, \dots, M$. Then we set

$$A_i = \bigcup_{l \text{ is odd}} A_i^l, \qquad A_{M+i} = \bigcup_{l \text{ is even}} A_i^l, \qquad i = 1, \dots, M.$$

Then we constructed $N := 2M$ subsets which have the required properties. $\square$

Definition. Let P be a locally compact space and $\mathcal{S}$ be a σ-algebra of subsets of P. We say that μ is a **Radon measure** on $(P, \mathcal{S})$ if

(a) $\mathcal{S}$ contains all Borel subsets of P,

(b) $\mu(K) < \infty$ for every compact set $K \subset P$,

(c) $\mu(G) = \sup\{\mu(K);\ K \subset G \text{ is compact}\}$ for every open set $G \subset P$,

(d) $\mu(A) = \inf\{\mu(G);\ A \subset G, G \text{ is open}\}$ for every $A \in \mathcal{S}$,

(e) μ is complete.

Definition. Let μ be a measure on X. **Outer measure corresponding** to μ is defined by

$$\mu^*(A) = \inf\{\mu(B);\ A \subset B, B \text{ is } \mu\text{-measurable}\}.$$

Remark. Let μ be a Radon measure on $(\mathbf{R}^n, \mathcal{S})$ and $A \in \mathcal{S}$. Then there exist a Borel set $B \subset \mathbf{R}^n$ such that $A \subset B$ and $\mu(B \setminus A) = 0$. If ν is a Radon measure on $(\mathbf{R}^n, \mathcal{S}')$ with $\nu \ll \mu$, then $\mathcal{S} \subset \mathcal{S}'$.

Lemma 1.4. *Let μ be a measure on X and $\{A_j\}_{j=1}^\infty$ be an increasing sequence of subset of X. Then $\lim \mu^*(A_j) = \mu^*\big(\bigcup_{j=1}^\infty A_j\big)$.*

Theorem 1.5. *Let μ be a Radon measure on $\mathbf{R}^n$ and $\mathcal{F}$ be a system of closed balls in $\mathbf{R}^n$. Let A denote the set of centers of the balls in $\mathcal{F}$. Assume $\inf\{r;\ B(a,r) \in \mathcal{F}\} = 0$ for each $a \in A$. Then there exists a countable disjoint system $\mathcal{G} \subset \mathcal{F}$ such that $\mu\big(A \setminus \bigcup \mathcal{G}\big) = 0$.*

Proof. The case $\mu^*(A) < \infty$. Let N be the natural number from Theorem 1.3. Fix θ such that $1 - \frac{1}{N} < \theta < 1$.

Claim. Let $U \subset \mathbf{R}^n$ be an open set. There exists a disjoint finite system $\mathcal{H} \subset \mathcal{F}$ such that $\bigcup \mathcal{H} \subset U$ and

$$\mu^*\big((A \cap U) \setminus \bigcup \mathcal{H}\big) \le \theta \mu^*(A \cap U). \tag{1.5}$$

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Proof of Claim. We may assume that $\mu^*(A \cap U) > 0$. Let $\mathcal{F}_1 = \{B \in \mathcal{F};\ \operatorname{diam} B < 1, B \subset U\}$. By Theorem 1.3 there exist disjoint families $\mathcal{G}_1, \dots, \mathcal{G}_N \subset \mathcal{F}_1$ such that

$$A \cap U \subset \bigcup_{i=1}^{N} \bigcup \mathcal{G}_i.$$

Thus

$$\mu^*(A \cap U) \leq \sum_{i=1}^{N} \mu^*\big(A \cap U \cap \bigcup \mathcal{G}_i\big).$$

Consequently, there exists an integer $1 \leq j \leq N$ for which

$$\mu^*\big(A \cap U \cap \bigcup \mathcal{G}_j\big) \geq \frac{1}{N}\mu^*(A \cap U) > (1-\theta)\mu^*(A \cap U).$$

Using Lemma 1.4 we find a finite system $\mathcal{H} \subset \mathcal{G}_j$ such that

$$\mu^*\big(A \cap U \cap \bigcup \mathcal{H}\big) > (1-\theta)\mu^*(A \cap U).$$

The set $\bigcup \mathcal{H}$ is μ-measurable and therefore

$$\begin{aligned} \mu^*(A \cap U) &= \mu^*\big(A \cap U \cap \bigcup \mathcal{H}\big) + \mu^*\big(A \cap U \setminus \bigcup \mathcal{H}\big) \\ &\geq (1-\theta)\mu^*(A \cap U) + \mu^*\big(A \cap U \setminus \bigcup \mathcal{H}\big). \end{aligned}$$

This gives (1.5). □

Set $U_1 = \mathbf{R}^n$. Using Claim we find a disjoint finite system $\mathcal{H}_1 \subset \mathcal{F}$ such that $\bigcup \mathcal{H}_1 \subset U_1$ and

$$\mu^*\big((A \cap U_1) \setminus \bigcup \mathcal{H}_1\big) \leq \theta \mu^*(A \cap U_1).$$

Continuing by induction we obtain a sequence of open set (U_j) and finite disjoint finite systems $(\mathcal{H}_j)$ such that $U_{j+1} = U_j \setminus \bigcup \mathcal{H}_j$, $\mathcal{H}_j \subset \mathcal{F}$, $\bigcup \mathcal{H}_j \subset U_j$, and

$$\mu(A \cap U_{j+1}) = \mu^*\big((A \cap U_j) \setminus \bigcup \mathcal{H}_j\big) \leq \theta \mu^*(A \cap U_j)$$

for every $j \in \mathbf{N}$. Together we have

$$\mu^*\big(A \cap U_{j+1}\big) \leq \theta^j \mu^*(A)$$

for every $j \in \mathbf{N}$. Since $\mu^*(A) < \infty$ we get $\mu^*\big(A \setminus \bigcup_{j=1}^{\infty} \bigcup \mathcal{H}_j\big) = 0$. Thus we set $\mathcal{G} = \bigcup_{j=1}^{\infty} \mathcal{H}_j$ and we are done.

The general case. We find a sequence of bounded disjoint open sets $(G_j)_{j=1}^{\infty}$ such that $\mu\big(\mathbf{R}^n \setminus \bigcup_{j=1}^{\infty} G_j\big) = 0$. Then $\mu(G_j) < \infty$ for every $j \in \mathbf{N}$ and we proceed as in the proof of Theorem 1.1 □

1.2 Differentiation of measures

Notation. The symbol $\mathcal{B}$ stands for the family of all closed balls in $\mathbf{R}^n$.

Definition. Let ν and μ are measures on $\mathbf{R}^n$ and $x \in \mathbf{R}^n$. Then we define

- **upper derivative of ν with respect to μ at x** by

$$\overline{D}(\nu, \mu, x) = \lim_{r \to 0+} \left(\sup\{\nu(B)/\mu(B);\ x \in B,\ B \in \mathcal{B},\ \operatorname{diam} B < r\}\right),$$

  if the term at the right side is defined,

- **lower derivative of ν with respect to μ at x** by

$$\underline{D}(\nu, \mu, x) = \lim_{r \to 0+} \left(\inf\{\nu(B)/\mu(B);\ x \in B,\ B \in \mathcal{B},\ \operatorname{diam} B < r\}\right),$$

  if the term at the right side is defined,

- **derivative of ν with respect to μ at x** (denoting $D(\nu, \mu, x)$) as the common value of $\overline{D}(\nu, \mu, x)$ and $\underline{D}(\nu, \mu, x)$, if it is defined.

Remark. The value $\overline{D}(\nu, \mu, x)$ ($\underline{D}(\nu, \mu, x)$) is well defined if and only if

$$\forall B \in \mathcal{B},\ x \in B \colon \mu(B) > 0.$$

Theorem 1.6. *Let ν and μ be Radon measures on $\mathbf{R}^n$ and μ satisfy Vitali theorem. Then $\overline{D}(\nu, \mu, x)$ and $\underline{D}(\nu, \mu, x)$ exist μ-a.e.*

Proof. Denote

$$\begin{aligned} M &= \{x \in \mathbf{R}^n;\ \overline{D}(\nu, \mu, x) \text{ is not defined}\}, \\ \mathcal{V} &= \{B \in \mathcal{B};\ \mu(B) = 0\}. \end{aligned}$$

The family $\mathcal{V}$ is a Vitali cover of M. We find a countable disjoint system $\mathcal{A} \subset \mathcal{V}$ such that $\mu(M \setminus \bigcup \mathcal{A}) = 0$. The we have

$$\mu\left(\bigcup \mathcal{A}\right) = \sum_{B \in \mathcal{A}} \mu(B) = 0,$$

therefore $\mu(M) = 0$.

The proof for $\underline{D}(\nu, \mu, x)$ is analogous. □

Theorem 1.7. *Let ν and μ be Radon measures on $\mathbf{R}^n$, μ satisfy Vitali theorem, $c \in (0, \infty)$, and $M \subset \mathbf{R}^n$.*

(i) *If for every $x \in M$ we have $\overline{D}(\nu, \mu, x) > c$, then $\nu^*(M) \geq c\mu^*(M)$.*

(ii) *If for every $x \in M$ we have $\underline{D}(\nu,\mu,x) < c$, then there exists $H \subset M$ such that $\mu(M \setminus H) = 0$ and $\nu^*(H) \le c\mu^*(M)$.*

Proof. (i) Choose $\varepsilon > 0$. There exists an open set $G \subset \mathbf{R}^n$ with $M \subset G$ and $\nu(G) \le \nu^*(M) + \varepsilon$. Set

$$\mathcal{V} = \{B \in \mathcal{B};\ B \subset G, \nu(B) > c\mu(B)\}.$$

The family $\mathcal{V}$ is a Vitali cover of M. There exists a disjoint countable subfamily $\mathcal{A} \subset \mathcal{V}$ with $\mu(M \setminus \bigcup \mathcal{A}) = 0$. Then we have

$$\begin{aligned}\nu^*(M) + \varepsilon \ge \nu(G) \ge \nu\big(\bigcup \mathcal{A}\big) &= \sum_{B\in\mathcal{A}} \nu(B)\\ &\ge \sum_{B\in\mathcal{A}} c\mu(B) = c\mu\big(\bigcup \mathcal{A}\big) \ge c\mu^*(M).\end{aligned}$$

Taking $\varepsilon \to 0+$ we get the desired inequality.

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(ii) Choose $k \in \mathbf{N}$. There exists an open set $G_k \subset \mathbf{R}^n$ such that $M \subset G_k$ and $\mu(G_k) \le \mu^*(M) + 1/k$. Set

$$\mathcal{V}_k = \{B \in \mathcal{B};\ B \subset G_k, \nu(B) < c\mu(B)\}.$$

The system $\mathcal{V}_k$ is a Vitali cover of M. Thus there exists a countable disjoint subfamily $\mathcal{A}_k \subset \mathcal{V}_k$ such that $\mu(M \setminus \bigcup \mathcal{A}_k) = 0$. Set $H_k = M \cap \bigcup \mathcal{A}_k$. Then $\mu(M \setminus H_k) = 0$, $H_k \subset M$ and we have

$$\begin{aligned}\nu^*(H_k) \le \nu\big(\bigcup \mathcal{A}_k\big) &= \sum_{B\in\mathcal{A}} \nu(B) \le c \sum_{B\in\mathcal{A}} \mu(B) = c\mu\big(\bigcup \mathcal{A}\big)\\ &\le c\mu(G_k) \le c\big(\mu^*(M) + \tfrac{1}{k}\big).\end{aligned}$$

Now we set $H = \bigcap_{k=1}^{\infty} H_k$. Then we have $\nu^*(H) \le c\mu^*(M)$ and

$$\mu(M \setminus H) = \mu^*(M \setminus H) \le \sum_{k=1}^{\infty} \mu^*(M \setminus H_k) = 0.$$

□

Theorem 1.8. *Let ν and μ be Radon measures on $\mathbf{R}^n$ and μ satisfies Vitali theorem. Then $D(\nu,\mu,x)$ is finite μ-a.e.*

Proof. Denote

$$\begin{aligned}D &= \{x \in \mathbf{R}^n;\ D(\nu,\mu,x) \in \langle 0,\infty)\},\\ N_1 &= \{x \in \mathbf{R}^n;\ \overline{D}(\nu,\mu,x) \text{ is not defined}\},\\ N_2 &= \{x \in \mathbf{R}^n;\ \underline{D}(\nu,\mu,x) \text{ is not defined}\},\\ N_3 &= \{x \in \mathbf{R}^n;\ \overline{D}(\nu,\mu,x) = \infty\},\\ N_4 &= \{x \in \mathbf{R}^n;\ \underline{D}(\nu,\mu,x) < \overline{D}(\nu,\mu,x)\}.\end{aligned}$$

Then we have

- $D = \mathbf{R}^n \setminus (N_1 \cup N_2 \cup N_3 \cup N_4)$,
- $\mu(N_1) = \mu(N_2) = 0$ (Theorem 1.6).

Further we define

$$\begin{aligned} A_k &= \{x \in \mathbf{R}^n;\ \overline{D}(\nu, \mu, x) > k\}, \\ A(r,s) &= \{x \in \mathbf{R}^n;\ \underline{D}(\nu, \mu, x) < s < r < \overline{D}(\nu, \mu, x)\}, \quad s, r \in \mathbf{Q}^+, s < r. \end{aligned}$$

The we have

$$\begin{aligned} N_3 &= \bigcap_{k=1}^{\infty} A_k, \\ N_4 &= \bigcup \{A(r,s);\ r, s \in \mathbf{Q}^+, s < r\}. \end{aligned}$$

We show $\mu(N_3) = 0$. Choose $Q \subset N_3$ bounded. By Theorem 1.7(i) we have

$$k\mu^*(Q) \le \nu^*(Q) < \infty$$

for every $k \in \mathbf{N}$. Therefore $\mu^*(Q) = 0$ and thus also $\mu(N_3) = 0$, since N_3 is a countable union of bounded sets.

We show $\mu(N_4) = 0$. It is sufficient to show $\mu(A(r,s)) = 0$ for every $s, r \in \mathbf{Q}^+, s < r$. Choose $Q \subset A(r,s)$ bounded. By Theorem 1.7(ii) there exists $H \subset Q$ such that $\mu(Q \setminus H) = 0$ and $\nu^*(H) \le s\mu^*(Q)$. By Theorem 1.7(i) we have $r\mu^*(H) \le \nu^*(H)$. We may conclude

$$r\mu^*(Q) = r\mu^*(H) \le \nu^*(H) \le s\mu^*(Q) < \infty.$$

Since $r > s > 0$, we have $\mu^*(Q) = 0$. This implies $\mu(A(r,s)) = 0$. □

Lemma 1.9. *Let ν and μ be Radon measures on $\mathbf{R}^n$ and μ satisfies Vitali theorem. Then the mappings $x \mapsto \overline{D}(\nu, \mu, x)$, $x \mapsto \underline{D}(\nu, \mu, x)$ are μ-measurable.*

Proof. We start with the following observation.

The set

$$M(r, \alpha) = \left\{x \in \mathbf{R}^n;\ \exists B \in \mathcal{B}\colon\ \operatorname{diam} B < r \wedge x \in B \wedge \tfrac{\nu(B)}{\mu(B)} < \alpha\right\}$$

is open for every $r > 0$ and $\alpha \in \mathbf{R}$.

If $x \in M(r, \alpha)$, then there exist $y \in \mathbf{R}^n$ and $s > 0$ with $x \in \overline{B}(y,s)$, $2s < r$,

$$\frac{\nu\big(\overline{B}(y,s)\big)}{\mu\big(\overline{B}(y,s)\big)} < \alpha.$$

We find $s' > s$ such that $2s' < r$, $\nu\big(\overline{B}(y,s')\big)/\mu\big(\overline{B}(y,s')\big) < \alpha$. Now we have $x \in B(y,s') \subset M(r,\alpha)$. This finishes the proof of the observation.

Denote $D = \{x \in \mathbf{R}^n;\ \underline{D}(\nu,\mu,x) \text{ exists finite}\}$. The set D is μ-measurable by Theorem 1.8. For every $x \in D$ we have

$$\begin{aligned}
&\underline{D}(\nu,\mu,x) < \alpha \\
&\Leftrightarrow \exists \tau \in \mathbf{Q}, \tau > 0\ \forall r \in \mathbf{Q}, r > 0\ \exists B \in B\colon\ \operatorname{diam} B < r,\ x \in B, \frac{\nu(B)}{\mu(B)} < \alpha - \tau \\
&\Leftrightarrow \exists \tau \in \mathbf{Q}, \tau > 0\ \forall r \in \mathbf{Q}, r > 0 :\ x \in M(r, \alpha - \tau).
\end{aligned}$$

The set $\{x \in \mathbf{R}^n;\ \underline{D}(\nu,\mu,x) < \alpha\}$ is intersection of D with a Borel set. This implies that the mapping $x \mapsto \underline{D}(\nu,\mu,x)$ is μ-measurable.

Measurability of the mapping $x \mapsto \overline{D}(\nu,\mu,x)$ can be proved analogously. □

Theorem 1.10. *Let ν and μ be Radon measures on $\mathbf{R}^n$, μ satisfies Vitali theorem, $\nu \ll \mu$, and $B \subset \mathbf{R}^n$ is μ-measurable. Then we have*

$$\int_B D(\nu,\mu,x)\, d\mu(x) = \nu(B).$$

Proof. Let $B \subset \mathbf{R}^n$ be a μ-measurable set. Choose $\beta \in \mathbf{R}$, $\beta > 1$. Define

$$\begin{aligned}
B_k &= \{x \in B;\ \beta^k < D(\nu,\mu,x) \le \beta^{k+1}\}, \qquad k \in \mathbf{Z}, \\
N &= \{x \in B;\ D(\nu,\mu,x) = 0\}.
\end{aligned}$$

These sets are μ-measurable by Lemma 1.9. Using Theorem 1.8 we have

$$\mu\Big(B \setminus \big(\bigcup_{k=-\infty}^{\infty} B_k \cup N\big)\Big) = 0.$$

Then we have

$$\begin{aligned}
\int_B D(\nu,\mu,x)\, d\mu(x) &= \sum_{k=-\infty}^{\infty} \int_{B_k} D(\nu,\mu,x)\, d\mu(x) \le \sum_{k=-\infty}^{\infty} \beta^{k+1} \mu(B_k) \\
&\le \sum_{k=-\infty}^{\infty} \beta^{k+1} \beta^{-k} \nu(B_k) \le \beta \nu(B).
\end{aligned}$$

Going $\beta \to 1+$ we get

$$\int_B D(\nu,\mu,x)\, d\mu(x) \le \nu(B).$$

Now let $\beta > 1$ again. Define

$$\begin{aligned}
B_k &= \{x \in B;\ \beta^k \le D(\nu,\mu,x) < \beta^{k+1}\}, \\
N &= \{x \in B;\ D(\nu,\mu,x) = 0\}.
\end{aligned}$$

Besides the equality

$$\mu\Big(B \setminus (\bigcup_{k=-\infty}^{\infty} B_k \cup N)\Big) = 0,$$

we have also $\nu(B \setminus (\bigcup_{k=-\infty}^{\infty} B_k \cup N)) = 0$, since $\nu \ll \mu$. By Theorem 1.7(ii) and absolute continuity of ν with respect to μ we obtain $\nu^*(Q) \le c\mu^*(Q) < \infty$ for any $c > 0$ and $Q \subset N$ bounded. Similarly as in the proof of Theorem 1.8 we get $\nu(N) = 0$. Then we have

$$\begin{aligned}\int_B D(\nu,\mu,x)\,d\mu(x) &\ge \sum_{k=-\infty}^{\infty} \int_{B_k} D(\nu,\mu,x)\,d\mu(x) \ge \sum_{k=-\infty}^{\infty} \beta^k \mu(B_k)\\ &\ge \sum_{k=-\infty}^{\infty} \beta^k \beta^{-(k+1)} \nu(B_k) = \frac{1}{\beta}\nu(B).\end{aligned}$$

Now it follows $\int_B D(\nu,\mu,x)\,d\mu(x) \ge \nu(B)$. □

The end of the lecture no. 6, 14. 11. 2022

1.3 Lebesgue points

Definition. Let μ be a Radon measure on $\mathbf{R}^n$. The symbol $\mathcal{L}^1_{loc}(\mu)$ denotes the set of all functions $f\colon \mathbf{R}^n \to \mathbf{C}$, which are μ-measurable and for every $x \in \mathbf{R}^n$ there exists $r > 0$ such that $\int_{B(x,r)} |f(t)|\,d\mu(t) < \infty$.

Definition. Let $f \in \mathcal{L}^1_{loc}(\mu)$. We say that $x \in \mathbf{R}^n$ is **Lebesgue point of f (with respect to μ)**, if it holds

$$\forall \varepsilon > 0\ \exists \delta > 0\ \forall B \in \mathcal{B}, x \in B, \operatorname{diam} B < \delta\colon \frac{\int_B |f(t) - f(x)|\,d\mu(t)}{\mu(B)} < \varepsilon.$$

Theorem 1.11. *Let μ be a Radon measure on $\mathbf{R}^n$ satisfying Vitali theorem and $f \in \mathcal{L}^1_{loc}(\mu)$. Then μ-a.e. points of f are Lebesgue points.*

Proof. Without any loss of generality we may assume that $\mu(\mathbf{R}^n) < \infty$ and $f \in \mathcal{L}^1(\mu)$. Let (C_k) be a sequence of closed discs in $\mathbf{C}$, which forms a basis of $\mathbf{C}$. We denote

$$g_k(x) := \operatorname{dist}(f(x), C_k), \qquad x \in \mathbf{R}^n.$$

The function g_k is nonnegative μ-measurable function satisfying $g_k \in \mathcal{L}^1(\mu)$. Let $\nu_k = \int g_k\,d\mu$. By Theorem 1.10 we have $D(\nu_k,\mu,x) = g_k(x)$ μ-a.e. Denote

$$P_k = \{x \in f^{-1}(C_k);\ \neg(D(\nu_k,\mu,x) = 0)\}.$$

We have $g_k = 0$ on $f^{-1}(C_k)$, therefore $\mu(P_k) = 0$. We show that every point from $\mathbf{R}^n \setminus \bigcup_{k=1}^{\infty} P_k$ is a Lebesgue point of f.

Let $x \in \mathbf{R}^n \setminus \bigcup_{k=1}^{\infty} P_k$. Choose $\varepsilon > 0$. We find C_k such that $f(x) \in C_k$ and $C_k \subset B(f(x), \varepsilon/2)$. For any $t \in \mathbf{R}^n$ it holds

$$|f(t) - f(x)| \leq g_k(t) + \varepsilon.$$

There exists $\delta > 0$ such that

$$\forall B \in \mathcal{B},\ x \in B,\ \operatorname{diam} B < \delta :\ \frac{\int_B g_k(t)\, d\mu(t)}{\mu(B)} < \varepsilon,$$

since $D(\nu_k, \mu, x) = 0$. Take $B \in \mathcal{B}$ with $x \in B$, $\operatorname{diam} B < \delta$ we get

$$\frac{\int_B |f(t) - f(x)|\, d\mu(t)}{\mu(B)} \leq \frac{\int_B g_k(t)\, d\mu(t) + \varepsilon\mu(B)}{\mu(B)} < 2\varepsilon.$$

This finishes the proof. □

1.4 Density theorem

Definition. Let μ be a measure on $\mathbf{R}^n$, $A \subset \mathbf{R}^n$ be μ-measurable, and $x \in \mathbf{R}^n$. We say that $c \in [0,1]$ is **μ-density of the set A at x**, if

$$\forall \varepsilon > 0\ \exists \delta > 0\ \forall B \in \mathcal{B},\ x \in B,\ \operatorname{diam} B < \delta \colon \Big|\frac{\mu(A \cap B)}{\mu(B)} - c\Big| < \varepsilon.$$

We denote $\mathrm{d}_\mu(A, x) = c$.

Theorem 1.12. *Let μ be a Radon measure on $\mathbf{R}^n$ satisfying Vitali theorem and $M \subset \mathbf{R}^n$ be μ-measurable. Then*

- $\mathrm{d}_\mu(M, x) = 1$ *for μ-a.e. $x \in M$,*
- $\mathrm{d}_\mu(M, x) = 0$ *for μ-a.e. $x \in \mathbf{R}^n \setminus M$.*

Proof. Define ν on $\mathbf{R}^n$ by

$$\nu(A) = \mu(A \cap M) \quad \text{for every } A \subset \mathbf{R}^n\ \mu\text{-measurable}.$$

Then we have

- $\mathrm{d}_\mu(M, x) = D(\nu, \mu, x)$, if at least one term is well defined,
- $\nu \ll \mu$,
- $\nu = \int \chi_M \, d\mu$.

By Theorem 1.10 we have $\nu = \int D(\nu, \mu, x)\, d\mu(x)$ therefore $\mathrm{d}_\mu(M, x) = D(\nu, \mu, x) = \chi_M(x)$ μ-a.e. □

1.5 AC and BV functions

Remark. For $a, c, b \in \mathbf{R}, a < c < b$, it holds

- $\mathrm{V}_a^b f = \mathrm{V}_a^c f + \mathrm{V}_c^b f$,
- $|f(b) - f(a)| \le \mathrm{V}_a^b f$.

Example. Let f be a function with continuous derivative on an interval $[a, b]$. Then $\mathrm{V}_a^b f = \int_a^b |f'(x)|\, dx$.

Remark. Let I be a closed nonempty interval. Then we have

(a) $f, g \in AC(I) \Rightarrow f + g \in AC(I)$,

(b) $f \in AC(I), \alpha \in \mathbf{R} \Rightarrow \alpha f \in AC(I)$.

Theorem 1.13. *Let $f \colon [a, b] \to \mathbf{R}$, $a < b$. Then f is absolutely continuous on $[a, b]$ if and only if f is difference of of two nondecreasing absolutely continuous functions on $[a, b]$.*

Proof. $\Rightarrow$ We denote $v(x) = \mathrm{V}_a^x f$, $x \in [a, b]$. For every $x, y \in I := [a, b]$, $x < y$, we have $v(y) - v(x) = V_x^y f$. The function v is well defined since $f \in BV([a, x]), x \in [a, b]$.

The function v is nondecreasing. This is obvious.

The function $v - f$ is nondecreasing. For every $x, y \in I, x < y$ we have

$$\big(v(y) - f(y)\big) - \big(v(x) - f(x)\big) = \big(v(y) - v(x)\big) - \big(f(y) - f(x)\big) = V_x^y f - \big(f(y) - f(x)\big) \ge 0.$$

The function v is absolutely continuous. Choose $\varepsilon > 0$. We find $\delta > 0$ such that

$$\sum_{j=1}^{m} |f(b_j) - f(a_j)| < \varepsilon,$$

whenever $a_1 < b_1 \le a_2 < b_2 \le \cdots \le a_m < b_m$ are points from $I = [a, b]$ with $\sum_{j=1}^{m}(b_j - a_j) < \delta$. Now assume that we have points $A_1 < B_1 \le A_2 < B_2 \le \cdots \le A_p < B_p$ from I satisfying $\sum_{j=1}^{p}(B_j - A_j) < \delta$. For each $j \in \{1, \dots, p\}$ we find points

$$A_j = a_1^j < b_1^j = a_2^j < b_2^j = \cdots < b_{m_j}^j = B_j$$

such that

$$v(B_j) - v(A_j) = V_{A_j}^{B_j} f < \sum_{i=1}^{m_j} |f(b_i^j) - f(a_i^j)| + \frac{\varepsilon}{p}.$$

The we have

$$\sum_{j=1}^{p} \sum_{i=1}^{m_j} (b_i^j - a_i^j) = \sum_{j=1}^{p} (B_j - A_j) < \delta$$

and
$$\sum_{j=1}^{p} |v(B_j) - v(A_j)| < \sum_{j=1}^{p} \Big(\sum_{i=1}^{m_j} |f(b_i^j) - f(a_i^j)| + \frac{\varepsilon}{p} \Big) < \varepsilon + \varepsilon = 2\varepsilon$$

Now we can write $f = v - (v - f)$. □

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Remark. Let $F\colon \mathbf{R} \to \mathbf{R}$ be nondecreasing function which is continuous at each point from the right. Then there exists a Radon measure ν_F such that F is the distribution function of ν_F, i.e.,

$$\nu_F\big((a, b]\big) = F(b) - F(a), \qquad a, b \in \mathbf{R}, a < b.$$

Lemma 1.14. *Let $f\colon (a, b) \to \mathbf{R}$, $x_0 \in (a, b)$, and $f'(x_0) \in \mathbf{R}$. Then we have*

$$\lim_{\substack{[x_1,x_2]\to[x_0,x_0]\\ x_1\le x_0\le x_2, x_1\ne x_2}} \frac{f(x_2) - f(x_1)}{x_2 - x_1} = f'(x_0).$$

Lemma 1.15. *Let $f\colon (a, b) \to \mathbf{R}$ be nondecreasing on (a, b), $C(f)$ be the set of all points of continuity of f, and $A \in \mathbf{R}$. Then for every $x_0 \in C(f)$ it holds*

$$f'(x_0) = A \Leftrightarrow \lim_{\substack{[x_1,x_2]\to[x_0,x_0]\\ x_1\le x_0\le x_2, x_1\ne x_2\\ x_1,x_2\in C(f)}} \frac{f(x_2) - f(x_1)}{x_2 - x_1} = A.$$

Lemma 1.16. *Let f be a distribution function of a measure μ on $\mathbf{R}$, $x_0 \in C(f)$, $A \in \mathbf{R}$. Then*

$$f'(x_0) = A \Leftrightarrow D(\mu, \lambda_1, x_0) = A.$$

Theorem 1.17 (Lebesgue)**.** *Let f be a monotone function on an interval I. Then we have*

- *$f'(x)$ exists a.e. in I,*
- *f' is measurable and $\left|\int_a^b f'\right| \le |f(b) - f(a)|$, whenever $a, b \in I, a < b$,*
- *$f' \in \mathcal{L}^1_{loc}(I)$.*

Theorem 1.18. *Let I be a nonempty interval and $f \in BV(I)$. Then $f'(x)$ exists finite a.e. in I.*

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Theorem 1.19. *Let $f\colon [a, b] \to \mathbf{R}$, $a < b$. Then the following assertions are equivalent.*

(i) $f \in AC([a, b])$.

(ii) *We have* $\varphi \in \mathcal{L}^1([a,b])$ *such that*

$$f(x) = f(a) + \int_a^x \varphi(t)\,dt, \qquad x \in [a,b].$$

(iii) $f'(x)$ *exists a.e. in* $[a,b]$, $f' \in \mathcal{L}^1([a,b])$ *and*

$$f(x) = f(a) + \int_a^x f'(t)\,dt, \qquad x \in [a,b].$$

Theorem 1.20 (per partes for Lebesgue integral)**.** *Let* $f, g \in AC([a,b])$. *Then we have*

$$\int_a^b f'g = [fg]_a^b - \int_a^b fg'.$$

Theorem 1.21. *Let* g *be a nonnegative function on* $[a,b]$ *with* $g \in \mathcal{L}^1([a,b])$. *Let* f *be a continuous function on* $[a,b]$. *The there exists* $\xi \in [a,b]$ *such that*

$$\int_a^b fg = f(\xi)\int_a^b g.$$

Theorem 1.22. *Let* $f \in \mathcal{L}^1([a,b])$ *and* g *be a monotone function on* $[a,b]$. *Then there exists* $\xi \in [a,b]$ *such that*

$$\int_a^b fg = g(a)\int_a^\xi f + g(b)\int_\xi^b f.$$

1.6 Rademacher theorem

Definition. Let $M \subset \mathbf{R}^n$. We say that $f\colon M \to \mathbf{R}$ is **Lipschitz (on** M**)**, if there exists $K > 0$ such that

$$\forall x, y \in M\colon |f(x) - f(y)| \leq K||x - y||.$$

Remark. If f is Lipschitz on M, then f is continuous on M.

Theorem 1.23. *Let* $G \subset \mathbf{R}^n$ *be open nonempty and* $f\colon G \to \mathbf{R}$ *be Lipschitz on* G. *Then* f *is differentiable a.e. on* G.

Lemma 1.24. *Let* $f\colon \mathbf{R}^n \to \mathbf{R}$ *be continuous and* $i \in \{1, \dots, n\}$. *Then the set*

$$D_i = \left\{x \in \mathbf{R}^n;\ \tfrac{\partial f}{\partial x_i}(x) \text{ exists}\right\}$$

is Borel.

Proof. We have

$$
\begin{aligned}
&\frac{\partial f}{\partial x_i}(x) \text{ exists}\\
&\Leftrightarrow \forall\varepsilon > 0\ \exists\delta > 0\ \forall t_1, t_2 \in (-\delta, \delta) \setminus \{0\} \colon \left|\tfrac{f(x+t_1e_i)-f(x)}{t_1} - \tfrac{f(x+t_2e_i)-f(x)}{t_2}\right| < \varepsilon\\
&\Leftrightarrow \forall\varepsilon \in \mathbf{Q}^+\ \exists\delta \in \mathbf{Q}^+\ \forall t_1, t_2 \in \big((-\delta, \delta) \cap \mathbf{Q}\big) \setminus \{0\} \colon \left|\tfrac{f(x+t_1e_i)-f(x)}{t_1} - \tfrac{f(x+t_2e_i)-f(x)}{t_2}\right| < \varepsilon.
\end{aligned}
$$

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For $\varepsilon > 0$ and nonzero t_1, t_2 denote

$$
D(\varepsilon, t_1, t_2) = \Big\{x \in \mathbf{R}^n;\ \left|\tfrac{f(x+t_1e_i)-f(x)}{t_1} - \tfrac{f(x+t_2e_i)-f(x)}{t_2}\right| < \varepsilon\Big\}.
$$

The set $D(\varepsilon, t_1, t_2)$ is open since f is continuous. We have

$$
D_i = \bigcap_{\varepsilon\in\mathbf{Q}^+} \bigcup_{\delta\in\mathbf{Q}^+} \bigcap_{\substack{t_1\in(-\delta,\delta)\cap\mathbf{Q}\\ t_1\neq 0}} \bigcap_{\substack{t_2\in(-\delta,\delta)\cap\mathbf{Q}\\ t_2\neq 0}} D(\varepsilon, t_1, t_2),
$$

therefore D_i is Borel. $\square$

Lemma 1.25. *Let $\beta > 0$, $A \neq \emptyset$, $f_\alpha, \alpha \in A$, be β-Lipschitz function on $\mathbf{R}^n$ and $x \in \mathbf{R}^n$ be such that $\sup_{\alpha\in A} f_\alpha(x)$ is finite. Then the function $z \mapsto \sup_{\alpha\in A} f_\alpha(z)$ is β-Lipschitz on $\mathbf{R}^n$.*

Proof. Let $u, v \in \mathbf{R}^n$. Then $|f_\gamma(u) - f_\gamma(x)| \leq \beta||u - x||$ for any $\gamma \in A$, therefore

$$
f_\gamma(u) \leq f_\gamma(x) + \beta||u - x|| \leq \sup_{\alpha\in A} f_\alpha(x) + \beta||u - x||.
$$

This implies

$$
\sup_{\gamma\in A} f_\gamma(u) \leq \sup_{\alpha\in A} f_\alpha(x) + \beta||u - x||,
$$

thus $\sup_{\gamma\in A} f_\gamma(u) \in \mathbf{R}$. Further we have

$$
f_\gamma(u) \leq f_\gamma(v) + \beta||u - v|| \leq \sup_{\alpha\in A} f_\alpha(v) + \beta||u - v|| \quad \text{for every } \gamma \in A.
$$

We get

$$
\sup_{\gamma\in A} f_\gamma(u) \leq \sup_{\alpha\in A} f_\alpha(v) + \beta||u - v||.
$$

Thus we have

$$
\sup_{\alpha\in A} f_\alpha(u) - \sup_{\alpha\in A} f_\alpha(v) \leq \beta||u - v||.
$$

Interchanging the roles of u and v we obtain

$$
\sup_{\alpha\in A} f_\alpha(v) - \sup_{\alpha\in A} f_\alpha(u) \leq \beta||u - v||,
$$

which proves β-Lipschitzness. $\square$

Lemma 1.26. *Let $E \subset \mathbf{R}^n$ be nonempty and $f\colon E \to \mathbf{R}$ be β-Lipschitz. Then there exists β-Lipschitz function $\tilde{f}\colon \mathbf{R}^n \to \mathbf{R}$ with $\tilde{f}|_E = f$.*

Proof. The function $f_x\colon y \mapsto f(x) - \beta \cdot ||y-x||$ is β-Lipschitz for every $x \in E$ since

$$|f_x(u) - f_x(v)| = \big|\beta \cdot ||u-x|| - \beta \cdot ||v-x||\big| \le \beta ||u-v||$$

for every $u, v \in \mathbf{R}^n$. For every $y \in E$ we have $\sup_{x\in E} f_x(y) \le f(y)$. Using Lemma 1.25 we get the mapping defined by

$$\tilde{f}(y) = \sup_{x\in E}(f(x) - \beta||y-x||)$$

is β-Lipschitz on $\mathbf{R}^n$. For $z \in E$ we have $\tilde{f}(z) \ge f_z(z) = f(z)$. Moreover $f_x(z) = f(x) - \beta||z-x|| \le f(z)$, which gives $\tilde{f}(z) \le f(z)$. Thus we prove $\tilde{f}(z) = f(z)$. $\square$

Proof of Theorem 1.23. By Lemma 1.26 we may suppose that f is Lipschitz with the constant β on $\mathbf{R}^n$, i.e.,

$$\forall x, y \in \mathbf{R}^n\colon |f(x) - f(y)| \le \beta||x-y||.$$

We show that f is differentiable a.e. This gives also the statement of the theorem. Let $E \subset \mathbf{R}^n$ be a set of those points where at least one partial derivative does not exist. The set $\mathbf{R}^n \setminus D_i$ is by Lemma 1.24 measurable. We use Fubini theorem and Rademacher theorem for $n = 1$ (see Remark) to get $\lambda_n(\mathbf{R}^n \setminus D_i) = 0$. Then we have $\lambda_n(E) = 0$, since $E = \bigcup_{i=1}^n (\mathbf{R}^n \setminus D_i)$.

For $p, q \in \mathbf{Q}^n$, $m \in \mathbf{N}$, denote

$$S(p,q,m) = \Big\{x \in \mathbf{R}^n;\ \forall i \in \{1,\dots,n\}\ \forall t \in (-1/m, 1/m) \setminus \{0\}\colon p_i \le \tfrac{f(x+te_i)-f(x)}{t} \le q_i\Big\}.$$

It is easy to verify that the set $S(p,q,m)$ is Borel. Let $\tilde{S}(p,q,m)$ be the set of all points of $S(p,q,m)$, where $S(p,q,m)$ has density 1. Then Theorem 1.12 gives

$$\lambda_n\big(S(p,q,m) \setminus \tilde{S}(p,q,m)\big) = 0.$$

The set

$$N = \bigcup \{S(p,q,m) \setminus \tilde{S}(p,q,m);\ p, q \in \mathbf{Q}^n, m \in \mathbf{N}\}$$

is of measure zero.

We show that f is differentiable at each point $x \in \mathbf{R}^n \setminus (E \cup N)$. Take $x \in \mathbf{R}^n \setminus (E \cup N)$ and $\varepsilon \in (0,1)$. Choose $p, q \in \mathbf{Q}^n$ such that

$$q_i - \varepsilon < p_i < \frac{\partial f}{\partial x_i}(x) < q_i, \quad i = 1, \dots, n.$$

Then there is $m \in \mathbf{N}$ such that $x \in S(p,q,m)$. Since $x \notin N$, the point x is a point of density of the set $S(p,q,m)$. Denote $S = S(p,q,m)$.

We find $\delta \in (0, 1/m)$ such that

$$\lambda_n\big(B(x,r) \setminus S\big) \le \left(\tfrac{\varepsilon}{2}\right)^n \lambda_n(B(x,r))$$

for every $r \in (0, 2\delta)$. Notice that the set $B(x, (1+\varepsilon)\tau) \setminus S$ does not contain a ball with radius $\varepsilon\tau$, whenever $\tau \in (0, \delta)$. Otherwise it would hold

$$c_n(\varepsilon\tau)^n \leq (\varepsilon/2)^n c_n (1+\varepsilon)^n \tau^n,$$

a contradiction. (The symbol c_n denotes n-dimensional measure of the unit ball.)

Choose $y \in B(x, \delta)$, $y \neq x$. Denote

$$y^i = [y_1, y_2, \ldots, y_i, x_{i+1}, \ldots, x_n].$$

For every $i \in \{0, \ldots, n\}$ define a ball $B_i = B(y^i, \varepsilon||y - x||)$. Using the preceding observation we have $B_i \cap S \neq \emptyset$. Find points $z^i \in S \cap B_i$, $i = 0, \ldots, n-1$, and denote $w^i = z^{i-1} + (y_i - x_i)e_i$, $i = 1, \ldots, n$.

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Then we have

$$\begin{aligned} p_i &\leq \frac{f(w^i) - f(z^{i-1})}{y_i - x_i} \leq q_i \quad \text{if } x_i \neq y_i, \\ p_i &< \frac{\partial f}{\partial x_i}(x) < q_i, \end{aligned}$$

therefore

$$\left| f(w^i) - f(z^{i-1}) - \frac{\partial f}{\partial x_i}(x)(y_i - x_i) \right| \leq (q_i - p_i)|y_i - x_i| \leq \varepsilon \|y - x\|.$$

Then we have

$$\begin{aligned} &\left| f(y) - f(x) - \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(x)(y_i - x_i) \right| \\ &\leq \sum_{i=1}^{n} \left| f(w^i) - f(z^{i-1}) - \frac{\partial f}{\partial x_i}(x)(y_i - x_i) \right| + \sum_{i=1}^{n} (|f(y^i) - f(w^i)| + |f(z^{i-1}) - f(y^{i-1})|) \\ &\leq n\varepsilon||y - x|| + 2n\beta\varepsilon||y - x|| = \varepsilon(n + 2n\beta)||y - x||, \end{aligned}$$

thus the proof is finished. □

Remark. Let us mention the following two deep results of D. Preiss ([2]).

1. Let H be a Hilbert space and $f \colon H \to \mathbf{R}$ be Lipschitz. Then there exists $x \in H$, where f is *Fréchet differentiable*, i.e., there exists a continuous linear mapping $L \colon H \to \mathbf{R}$ such that
$$\lim_{h \to 0} \frac{|f(x+h) - f(x) - L(h)|}{||h||} = 0.$$
2. There exists a closed measure zero set $F \subset \mathbf{R}^2$ such that any Lipschitz function on $\mathbf{R}^2$ is differentiable at some point of F.

1.7 Maximal operator

Definition. Let $f\colon \mathbf{R}^n \to \mathbf{R}$ be measurable. For $x \in \mathbf{R}^n$ we define

$$Mf(x) = \sup_{B \in \mathcal{B}, x \in B} \frac{1}{\lambda_n(B)} \int_B |f|.$$

Lecture no. 3

Theorem 1.27 (Hardy-Littlewood-Wiener)**.**

(a) *If $f \in L^p(\mathbf{R}^n)$, $1 \le p \le \infty$, then Mf is finite a.e.*

(b) *There exists $c > 0$ such that for every $f \in L^1(\mathbf{R}^n)$ and $\alpha > 0$ we have*

$$\lambda_n(\{x \in \mathbf{R}^n;\ Mf(x) > \alpha\}) \le \frac{c}{\alpha}\|f\|_1.$$

(c) *Let $p \in (1, \infty]$. Then there exists A such that for every $f \in L^p(\mathbf{R}^n)$ we have $\|Mf\|_p \le A\|f\|_p$.*

1.8 Lipschitz functions and $W^{1,\infty}$

Remark. We have

$$W^{1,\infty}(\Omega) = L^p(\Omega) \cap \big\{u;\ \partial_i u \in L^\infty(\Omega) \text{ (in the sense of distributions)}, i \in \{1, \dots, n\}\big\}.$$

Theorem 1.28. *Let $U \subset \mathbf{R}^n$ be open. Then $f\colon U \to \mathbf{R}$ is local Lipschitz on U if and only if $f \in W^{1,\infty}_{\mathrm{loc}}(U)$.*

Without proof.

Chapter 2

Hausdorff measures

2.1 Basic notions

Convention. We will assume that (P, ρ) is a metric space.

Definition. Let $p > 0$, $A \subset P$. Denote

$$\begin{aligned}\mathcal{H}_p(A,\delta) &= \inf\Big\{\sum_{j=1}^{\infty}(\operatorname{diam} A_j)^p;\ A \subset \bigcup_{j=1}^{\infty} A_j,\ \operatorname{diam} A_j \le \delta\Big\}, \qquad \delta > 0;\\ \mathcal{H}_p(A) &= \sup_{\delta>0}\mathcal{H}_p(A,\delta).\end{aligned}$$

The function $A \mapsto \mathcal{H}_p(A)$ is called **p-dimensional outer Hausdorff measure**.

Remark. Definice $\mathcal{H}_s$ se nezmění, pokud budeme uvažovat A_n uzavřené (resp. otevřené).

Definition. Outer measure γ on P is called **metric**, if for every $A, B \subset P$ with $\inf\{\rho(x,y);\ x \in A, y \in B\} > 0$ we have $\gamma(A \cup B) = \gamma(A) + \gamma(B)$.

Theorem 2.1. *Let γ be a metric outer measure on P. Then every Borel subset of P is γ-measurable.*

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Theorem 2.2. *$\mathcal{H}_p$ is a metric outer measure.*

Corollary 2.3. *Every Borel subset of P is $\mathcal{H}_p$-measurable.*

Theorem 2.4. *Let $k, n \in \mathbf{N}$, $k \le n$, $K = [0,1)^k \times \{0\}^{n-k} \subset \mathbf{R}^n$. Then $0 < \mathcal{H}_k(K) < \infty$.*

Remark. It can be shown that $\kappa_k := \mathcal{H}_k\big([0,1]^k \times \{0\}^{n-k}\big) = (4/\pi)^{k/2}\Gamma(1+\frac{k}{2})$.

Definition. Let $k \in \mathbf{N}$. The k**-dimensional normalized Hausdorff measure** is defined by $H^k = \frac{1}{\kappa_k}\mathcal{H}_k$.

Theorem 2.5 (regularity of Hausdorff measure)**.** *Let $k, n \in \mathbf{N}, k \le n$, and $A \subset \mathbf{R}^n$. Then there exists a Borel set $B \subset \mathbf{R}^n$ such that $A \subset B$ and $H^k(A) = H^k(B)$.*

Theorem 2.6. *Let $n \in \mathbf{N}$ and $A \subset \mathbf{R}^n$. Then $H^n(A) = \lambda^{n*}(A)$.*

2.2 Area formula

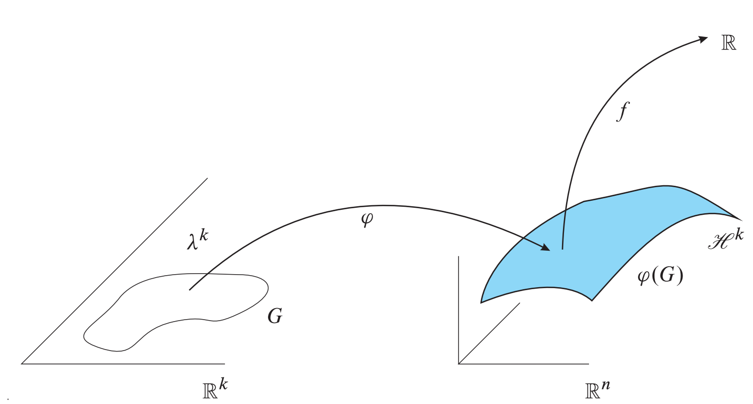
Notation. Let $k, n \in \mathbf{N}, k \leq n$, and $L\colon \mathbf{R}^k \to \mathbf{R}^n$ be a linear mapping. We denote $\operatorname{vol} L = \sqrt{\det L^T L}$.

Definition. Let $k, n \in \mathbf{N}$, $k \leq n$, and $G \subset \mathbf{R}^k$ be open. A mapping $f\colon G \to \mathbf{R}^n$ is said to be **regular**, if $f \in \mathcal{C}^1(G)$ and for every $x \in G$ the rank of $f'(a)$ is k.

Theorem 2.7 (area formula)**.** *Let $k, n \in \mathbf{N}, k \leq n$, $G \subset \mathbf{R}^k$ be an open set, $\varphi\colon G \to \mathbf{R}^n$ be an injective regular mapping and $f: \varphi(G) \to \mathbf{R}$ be H^k-measurable. Then we have*

$$\int_{\varphi(G)} f(x)\,\mathrm{d}\,H^k(x) = \int_G f(\varphi(t)) \operatorname{vol}\varphi'(t)\,\mathrm{d}\,\lambda^k(t),$$

if the integral at the right side converges.



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2.3 Hausdorff dimension

Lemma 2.8. *Let $0 < p < q$, $A \subset P$, and $\mathcal{H}_p(A) < \infty$. Then $\mathcal{H}_q(A) = 0$.*

Proof. Let $\delta \in (0, 1)$ and $\{A_j\}_{j=1}^{\infty}$ be a sequence of subsets of P such that $A \subset \bigcup_{j=1}^{\infty} A_j$, $\operatorname{diam} A_j \leq \delta$ for every $j \in \mathbf{N}$, and $\sum_{j=1}^{\infty} (\operatorname{diam} A_j)^p < \mathcal{H}_p(A) + 1$. Then we have

$$\begin{aligned}\mathcal{H}_q(A, \delta) &\leq \sum_{j=1}^{\infty} (\operatorname{diam} A_j)^q = \sum_{j=1}^{\infty} (\operatorname{diam} A_j)^p \cdot (\operatorname{diam} A_j)^{q-p} \\ &\leq \sum_{j=1}^{\infty} (\operatorname{diam} A_j)^p \cdot \delta^{q-p} \leq \delta^{q-p}(\mathcal{H}_p(A) + 1).\end{aligned}$$

Sending $\delta \to 0+$ we get $\mathcal{H}_q(A) = 0$. $\square$

Definition. Let $A \subset P$. **Hausdorff dimension** of A is defined by

$$\dim A = \inf\{t \geq 0;\ \mathcal{H}_t(A) < \infty\}.$$

Remark. By Lemma 2.8 we have

$$\mathcal{H}_t(A) = \begin{cases} \infty & \text{for } t < \dim(A), \\ 0 & \text{for } t > \dim(A). \end{cases}$$

Corollary 2.9. (i) *For every* $A \subset B \subset P$ *we have* $\dim A \leq \dim B$.

(ii) *For every* $A_i \subset P$, $i \in \mathbf{N}$, *we have* $\dim(\bigcup_{i=1}^{\infty} A_i) = \sup_i \dim A_i$.

(iii) *We have* $\dim([0,1]^k \times \{0\}^{n-k}) = k$, *in particular,* $\dim[0,1]^n = n$.

Example (Cantor set)**.** For $s \in \{\emptyset\} \cup \bigcup_{k=1}^{\infty}\{0,1\}^k$ we define inductively closed intervals I_s as follows

- $I_\emptyset = [0,1]$,
- if $I_s = [a,b]$, then $I_{s^\wedge i} = \begin{cases} [a, a + \frac{1}{3}(b-a)], & \text{if } i = 0, \\ [b - \frac{1}{3}(b-a), b], & \text{if } i = 1. \end{cases}$

Cantor set is defined by

$$C = \bigcap_{k=0}^{\infty} \bigcup_{s \in \{0,1\}^k} I_s.$$

The set C has the following properties:

- C is compact,
- C is nowhere dense,
- C is uncountable.

Theorem 2.10. *We have* $\dim C = \frac{\log 2}{\log 3}$.

Proof. Denote $d = \frac{\log 2}{\log 3}$.

We prove $\mathcal{H}_d(C) \leq 1$. We have $C \subset \bigcup_{s \in \{0,1\}^k} I_s$ and $\operatorname{diam} I_s \leq 3^{-k}$, $s \in \{0,1\}^k$. We infer

$$\sum_{s \in \{0,1\}^k} (\operatorname{diam} I_s)^d = 2^k \cdot (3^{-k})^d = 1.$$

Then we have $\mathcal{H}_d(C) \leq 1$.

We prove $\mathcal{H}_d(C) \geq 1/4$. It is sufficient to prove that

$$\sum_{j=1}^{\infty} (\operatorname{diam} I_j)^d \geq 1/4,$$

where $I_j, j \in \mathbf{N}$, are open intervals and $C \subset \bigcup_{j=1}^{\infty} I_j$. Convex envelope of an open set $G \subset \mathbf{R}$ is an open interval with the same diameter as G. The set C is compact, therefore there exist intervals $I_1, \dots, I_n$ covering C. Since C is nowhere dense, we may assume that, that the endpoints of $I_1, \dots, I_n$ are not in C. Then there exists $\delta > 0$ such that

$$\operatorname{dist}(C, \text{endpoints of } I_1, \dots, I_n) > \delta.$$

Let $k \in \mathbf{N}$ and $3^{-k} < \delta$. Then we have

$$\forall s \in \{0,1\}^k \, \exists j \in \{1, \dots, n\} \colon I_s \subset I_j. \tag{2.1}$$

Claim. Let $I \subset \mathbf{R}$ be an interval and $l \in \mathbf{N}$ we have

$$\sum_{\substack{I_s \subset I \\ s \in \{0,1\}^l}} (\operatorname{diam} I_s)^d \leq 4(\operatorname{diam} I)^d.$$

Proof of Claim. Suppose that the sum at the left side is nonzero. Let m be the smallest natural number such that I contains some $I_t, t \in \{0,1\}^m$. Then we have obviously $m \leq l$. Let $J_1, \dots, J_p$ are those intervals among $I_s,\ s \in \{0,1\}^m$, which intersect I. The we have $p \leq 4$ by the choice of m. Then we have

$$\begin{aligned} 4(\operatorname{diam} I)^d &\geq \sum_{i=1}^{p} (\operatorname{diam} J_i)^d = \sum_{i=1}^{p} \sum_{\substack{I_s \subset J_i \\ s \in \{0,1\}^l}} (\operatorname{diam} I_s)^d \\ &\geq \sum_{\substack{I_s \subset I \\ s \in \{0,1\}^l}} (\operatorname{diam} I_s)^d. \end{aligned}$$

Indeed, we have

$$\begin{aligned} (\operatorname{diam} J_i)^d &= (3^{-m})^d = 2^{-m}, \\ \sum_{\substack{I_s \subset J_i \\ s \in \{0,1\}^l}} (\operatorname{diam} I_s)^d &= 2^{l-m} \cdot (3^{-l})^d = 2^{-m}. \end{aligned}$$

□

Then we have

$$4 \sum_{j=1}^{\infty} (\operatorname{diam} I_j)^d \overset{\text{Claim}}{\geq} \sum_{j=1}^{n} \sum_{\substack{I_s \subset I_j \\ s \in \{0,1\}^k}} (\operatorname{diam} I_s)^d \overset{(2.1)}{\geq} \sum_{s \in \{0,1\}^k} (\operatorname{diam} I_s)^d = 1.$$

This finishes the proof. □

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Example. Let $\alpha > 0$. We define

$$E_\alpha = \left\{x \in \mathbf{R};\ \text{there exists infinitely many pairs } (p,q) \in \mathbf{Z} \times \mathbf{N} \text{ such that } \left|x - \tfrac{p}{q}\right| \leq q^{-(2+\alpha)}\right\}.$$

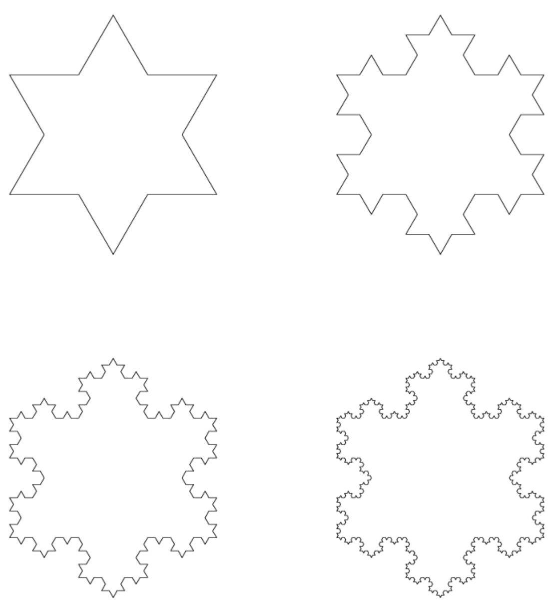
Jarník's theorem says that $\dim E_\alpha = \frac{2}{2+\alpha}$.

Definition. The mapping $f\colon \mathbf{R}^n \to \mathbf{R}^n$ is called **similitude with ratio** r if $\|f(x) - f(y)\| = r\|x - y\|$ for every $x, y \in \mathbf{R}^n$.

Theorem 2.11. *Let $m \in \mathbf{N}$ and $\psi_1, \dots, \psi_m$ be similitudes of $\mathbf{R}^n$ with ratios $r_1, \dots, r_m \in (0,1)$ such that there exists an open set $V \subset \mathbf{R}^n$ such that $\psi(V) \subset V$ and for every $i, j \in \{1, \dots, m\}, i \neq j$, we have $\psi_i(V) \cap \psi_j(V) = \emptyset$. Let E be a nonempty compact set satisfying $E = \bigcup_{i=1}^m \psi_i(E)$ and s satisfies $\sum_{i=1}^m r_i^s = 1$. Then we have $0 < \mathcal{H}_s(E) < \infty$.*

Without proof.

Example (Koch curve)**.** One can use Theorem 2.11 to prove Theorem 2.10 or to infer that Hausdorff dimension of Koch curve is $\frac{\log 4}{\log 3}$. Here we have several approximations of Koch curve.



Part II

Summer semester

Chapter 3

Area and coarea formulae

Theorem 3.1. *Let (P_1, ρ_1) and (P_2, ρ_2) be metric spaces, $s > 0$, and $f\colon P_1 \to P_2$ be β-Lipschitz. Then $\mathcal{H}_s\big(f(P_1)\big) \le \beta^s \mathcal{H}_s(P_1)$.*

Proof. Choose $\delta > 0$. Let sets $A_j, j \in \mathbf{N}$, satisfy $P_1 = \bigcup_{j=1}^{\infty} A_j$ and $\operatorname{diam} A_j < \delta$ for every $j \in \mathbf{N}$. Then we have $f(P_1) = \bigcup_{j=1}^{\infty} f(A_j)$ and $\operatorname{diam} f(A_j) \le \beta \operatorname{diam} A_j \le \beta\delta$. Then we have

$$\mathcal{H}_s(f(A), \beta\delta) \le \sum_{j=1}^{\infty} (\operatorname{diam} f(A_j))^s \le \sum_{j=1}^{\infty} \beta^s (\operatorname{diam} A_j)^s.$$

This implies $\mathcal{H}_s(f(A), \beta\delta) \le \beta^s \mathcal{H}_s(A, \delta)$. Sending $\delta \to 0+$, we get $\mathcal{H}_s(f(A)) \le \beta^s \mathcal{H}_s(A)$. □

Lemma 3.2. *Let $k, n \in \mathbf{N}, k \le n$, a $L\colon \mathbf{R}^k \to \mathbf{R}^n$ be an injective linear mapping. Then for every λ^k-measurable set $A \subset \mathbf{R}^k$ it holds*

$$H^k\big(L(A)\big) = \sqrt{\det L^T L} \cdot \lambda^k(A). \tag{3.1}$$

Proof. The mapping L is linear and injective, therefore the dimension of the vector space $L(\mathbf{R}^k)$ is k. Thus there exists a linear isometry $Q\colon \mathbf{R}^k \to \mathbf{R}^n$ such that $Q(\mathbf{R}^k) = L(\mathbf{R}^k)$. Then we have

$$\begin{aligned} H^k\big(L(A)\big) &= H^k(Q^{-1} \circ L(A)) = \lambda^k\big(Q^{-1} \circ L(A)\big) \\ &= |\det(Q^{-1}L)| \cdot \lambda^k(A). \end{aligned} \tag{3.2}$$

$$\begin{aligned} \big(\det(Q^{-1}L)\big)^2 &= \det\big((Q^{-1}L)^T Q^{-1}L\big) \\ &= \det\Big(\big(\langle Q^{-1}Le_i, Q^{-1}Le_j\rangle\big)_{i,j=1}^n\Big) \\ &= \det\Big(\big(\langle Le_i, Le_j\rangle\big)_{i,j=1}^n\Big) = \det(L^T L). \end{aligned} \tag{3.3}$$

The desired inequality (3.1) follows from (3.2) a (3.3). □

Notation. Let $k, n \in \mathbf{N}, k \le n$, and $L\colon \mathbf{R}^k \to \mathbf{R}^n$ be a linear mapping. We denote $\operatorname{vol} L = \sqrt{\det L^T L}$.

Remark. (a) The matrix $L^T L$ is called **Gram matrix**. By Lemmatu 3.2 we have $H^k\big(L([0,1]^k)\big) = \operatorname{vol} L$, thus $\operatorname{vol} L$ is k-dimensional volume of $L\big([0,1]^k\big)$. If $\varphi \in \mathcal{C}^1(G)$, then the mapping $t \mapsto \operatorname{vol} \varphi'(t)$ is continuous on the set G.

(b) If L is a matrix of the type $n \times k$, then the matrix $L^T L$ is symmetric and of the type $k \times k$.

(c) Gram determinant is nonnegative, since for every matrix A of the type $n \times k$ and for every $x \in \mathbf{R}^k$ we have $(A^T Ax, x) = (Ax, Ax) \geq 0$. Gram determinant is positive, whenever the rank of L is k.

Lemma 3.3. *Let $k, n \in \mathbf{N}, k \leq n$, $G \subset \mathbf{R}^k$ be open set, $\varphi\colon G \to \mathbf{R}^n$ be an injective regular mapping, $x \in G$, and $\beta > 1$. Then there exists a neighbourhood V of the point x such that*

(a) *the mapping $y \mapsto \varphi\big(\varphi'(x)^{-1}(y)\big)$ is β-Lipschitz on $\varphi'(x)(V)$,*

(b) *the mapping $z \mapsto \varphi'(x)\big(\varphi^{-1}(z)\big)$ is β-Lipschitz on $\varphi(V)$.*

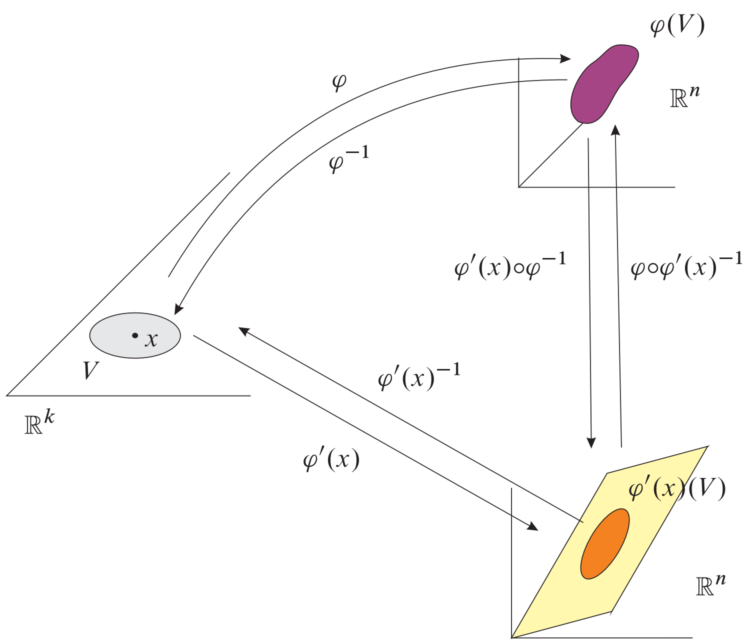


Figure 3.1:

Proof. First we infer several auxiliary inequalities. The linear mapping $v \mapsto \varphi'(x)(v)$ is injective, therefore there exists $\eta > 0$ such that

$$\forall v \in \mathbf{R}^k \colon \|\varphi'(x)(v)\| \geq \eta\|v\|. \tag{3.4}$$

We set $\eta = \inf\bigl\{\|\varphi'(x)(v)\|;\ v \in \mathbf{R}^k, \|v\| = 1\bigr\}$. The mapping $v \mapsto \varphi'(x)(v)$ is continuous and the unit sphere $\{v \in \mathbf{R}^k;\ \|v\| = 1\}$ is compact, therefore the infimum is attained at a point v_0. Since $\varphi'(x)(v_0) \neq 0$, η is positive.

We find $\varepsilon \in (0, \frac{1}{2}\eta)$ such that

$$\frac{2\varepsilon}{\eta} + 1 < \beta. \tag{3.5}$$

Further we find a ball V centered at the point x such that

$$\forall y \in V\colon \|\varphi'(y) - \varphi'(x)\| \leq \varepsilon.$$

We show that for every $u, v \in V$ it holds

$$\|\varphi(u) - \varphi(v) - \varphi'(x)(u - v)\| \leq \varepsilon\|u - v\|. \tag{3.6}$$

Fix $v \in V$ and consider the mapping

$$g\colon w \mapsto \varphi(w) - \varphi(v) - \varphi'(x)(w - v), \qquad w \in V.$$

For $w \in V$ we have $g'(w) = \varphi'(w) - \varphi'(x)$. Then we have

$$\begin{aligned}\|\varphi(u) - \varphi(v) - \varphi'(x)(u - v)\| &= \|g(u) - g(v)\| \\ &\leq \sup\{\|g'(w)\|;\ w \in V\} \cdot \|u - v\| \\ &\leq \varepsilon\|u - v\|,\end{aligned}$$

this implies (3.6).

Further we show that for every $u, v \in V$ we have

$$\|\varphi(u) - \varphi(v)\| \geq \frac{1}{2}\eta\|u - v\|. \tag{3.7}$$

For $u, v \in V$ we compute

$$\begin{aligned}\|\varphi(u) - \varphi(v)\| &\geq -\|\varphi(u) - \varphi(v) - \varphi'(x)(u - v)\| + \|\varphi'(x)(u - v)\| \\ &\geq -\varepsilon\|u - v\| + \eta\|u - v\| \geq \frac{1}{2}\eta\|u - v\|,\end{aligned}$$

this gives (3.7).

(a) Choose $a, b \in \varphi'(x)(V)$. We find $u, v \in V$ such that $\varphi'(x)(u) = a$, $\varphi'(x)(v) = b$. We compute

$$\begin{aligned}\bigl\|\varphi\bigl(\varphi'(x)^{-1}(a)\bigr) - \varphi\bigl(\varphi'(x)^{-1}(b)\bigr)\bigr\| &= \|\varphi(u) - \varphi(v)\| \\ &\leq \|\varphi(u) - \varphi(v) - \varphi'(x)(u - v)\| + \|\varphi'(x)(u - v)\| \\ &\overset{(3.6)}{\leq} \varepsilon\|u - v\| + \|\varphi'(x)(u - v)\| \\ &\overset{(3.4)}{\leq} \frac{\varepsilon}{\eta}\|a - b\| + \|a - b\| = \bigl(\tfrac{\varepsilon}{\eta} + 1\bigr)\|a - b\| \\ &\overset{(3.5)}{\leq} \beta\|a - b\|.\end{aligned}$$

(b) Choose $p, q \in \varphi(V)$. We find $u, v \in V$ with $\varphi(u) = p$, $\varphi(v) = q$. Compute

$$\begin{aligned}
\|\varphi'(x)\big(\varphi^{-1}(p)\big) - \varphi'(x)\big(\varphi^{-1}(q)\big)\| &= \|\varphi'(x)(u) - \varphi'(x)(v)\| \\
&= \|\varphi'(x)(u - v)\| \\
&\le \|\varphi(u) - \varphi(v) - \varphi'(x)(u - v)\| + \|\varphi(u) - \varphi(v)\| \\
&\overset{(3.6)}{\le} \varepsilon\|u - v\| + \|p - q\| \\
&\overset{(3.7)}{\le} \tfrac{2\varepsilon}{\eta}\|\varphi(u) - \varphi(v)\| + \|p - q\| = (\tfrac{2\varepsilon}{\eta} + 1)\|p - q\| \\
&\overset{(3.5)}{\le} \beta\|p - q\|.
\end{aligned}$$

This finishes the proof. □

The end of the lecture no. 1, 13. 2. 2023

Lemma 3.4. *Let $k, n \in \mathbf{N}, k \le n$, $G \subset \mathbf{R}^k$ be an open set, $\varphi\colon G \to \mathbf{R}^n$ be an injective regular mapping, $x \in G$ a $\alpha > 1$. Then there exists a neighbourhood V of x such that for every λ^k-measurable $E \subset V$ we have*

$$\alpha^{-1}\int_E \operatorname{vol}\varphi'(t)\,\mathrm{d}\,\lambda^k(t) \le H^k\big(\varphi(E)\big) \le \alpha\int_E \operatorname{vol}\varphi'(t)\,\mathrm{d}\,\lambda^k(t).$$

Proof. Find $\beta > 1$ a $\tau > 1$ such that

$$\beta^k\tau < \alpha. \tag{3.8}$$

By Lemma 3.3 we find V_1 of x such that for φ and β (a) and (b) of the lemma holds. Using continuity of the mapping $t \mapsto \operatorname{vol}\varphi'(t)$ on G we find a neighbourhood V_2 of x such that

$$\forall t \in V_2\colon \tau^{-1}\operatorname{vol}\varphi'(x) \le \operatorname{vol}\varphi'(t) \le \tau\operatorname{vol}\varphi'(x). \tag{3.9}$$

Set $V = V_1 \cap V_2$. We show that V is the desired neighbourhood.

Let $E \subset V$ be λ^k-measurable. By (3.9) we get

$$\tau^{-1}\operatorname{vol}\varphi'(x)\cdot\lambda^k(E) \le \int_E \operatorname{vol}\varphi'(t)\,\mathrm{d}\,\lambda^k(t) \le \tau\operatorname{vol}\varphi'(x)\cdot\lambda^k(E). \tag{3.10}$$

By Lemma 3.2 we have $\operatorname{vol}\varphi'(x)\cdot\lambda^k(E) = H^k\big(\varphi'(x)(E)\big)$, and we can write

$$\tau^{-1}H^k\big(\varphi'(x)(E)\big) \le \int_E \operatorname{vol}\varphi'(t)\,\mathrm{d}\,\lambda^k(t) \le \tau H^k\big(\varphi'(x)(E)\big). \tag{3.11}$$

By Lemma 3.3(a) and by the choice of V_1 we get

$$\begin{aligned}
H^k\big(\varphi(E)\big) &= H^k\big(\varphi\circ\varphi'(x)^{-1}\circ\varphi'(x)(E)\big) \le \beta^k H^k\big(\varphi'(x)(E)\big) \\
&\overset{(3.11)}{\le} \beta^k\tau\int_E \operatorname{vol}\varphi'(t)\,\mathrm{d}\,\lambda^k(t) \overset{(3.8)}{\le} \alpha\int_E \operatorname{vol}\varphi'(t)\,\mathrm{d}\,\lambda^k(t).
\end{aligned}$$

By Lemma 3.3(b) and by the choice of V_1 we get

$$H^k\big(\varphi(E)\big) \geq \beta^{-k} H^k\big(\varphi'(x) \circ \varphi^{-1} \circ \varphi(E)\big) = \beta^{-k} H^k\big(\varphi'(x)(E)\big)$$
$$\overset{(3.11)}{\geq} \beta^{-k}\tau^{-1} \int_E \operatorname{vol} \varphi'(t) \,\mathrm{d}\, \lambda^k(t) \overset{(3.8)}{\geq} \alpha^{-1} \int_E \operatorname{vol} \varphi'(t) \,\mathrm{d}\, \lambda^k(t).$$

□

Theorem 3.5 (area formula)**.** *Let $k, n \in \mathbf{N}, k \leq n$, $G \subset \mathbf{R}^k$ be an open set, $\varphi\colon G \to \mathbf{R}^n$ be an injective regular mapping and $f : \varphi(G) \to \mathbf{R}$ be H^k-measurable. Then we have*

$$\int_{\varphi(G)} f(x) \,\mathrm{d}\, H^k(x) = \int_G f(\varphi(t)) \operatorname{vol} \varphi'(t) \,\mathrm{d}\, \lambda^k(t),$$

if the integral at the right side converges.

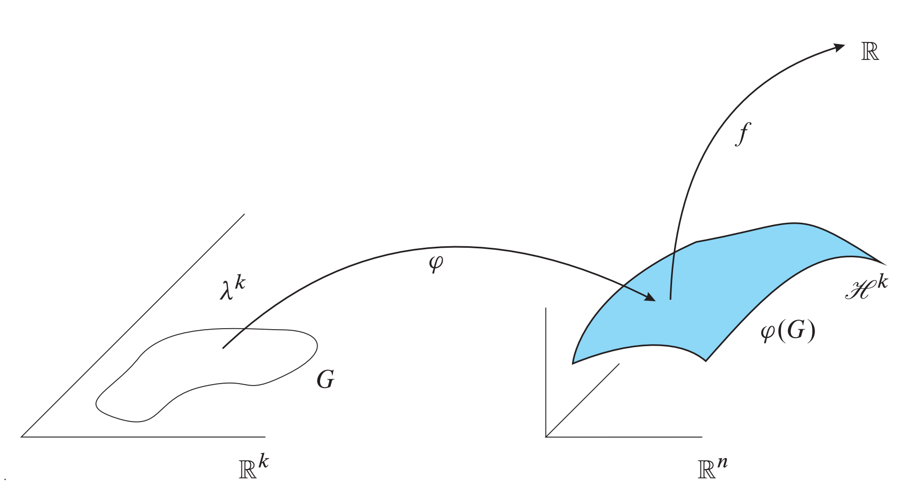


Figure 3.2:

Proof. The mapping φ is injective, therefore there exists an inverse mapping φ^{-1}. Each open set $H \subset G$ is a countable union of compact sets, therefore $\varphi(H)$ is a countable union of compact sets. Thus we get that φ^{-1} is Borel and the set $\varphi(G)$ is Borel.

The mappings φ, φ^{-1} are locally Lipschitz by Lemma 3.3. Therefore $\varphi(G)$ is H^k-σ-finite by Theorem 3.1.

1. Suppose that $f = \chi_L$, where $L \subset \varphi(G)$ is H^k-measurable. We show

$$H^k(L) = \int_{\varphi^{-1}(L)} \operatorname{vol} \varphi'(t) \,\mathrm{d}\, \lambda^k(t). \tag{3.12}$$

Choose $\alpha > 1$. By Lemma 3.4 we find for every $y \in G$ a neighbourhood $V_y \subset G$ of the point y such that for every λ^k-measurable set $E \subset V_y$ we have

$$\alpha^{-1} \int_E \operatorname{vol} \varphi'(t) \,\mathrm{d}\,\lambda^k(t) \leq H^k(\varphi(E)) \leq \alpha \int_E \operatorname{vol} \varphi'(t) \,\mathrm{d}\,\lambda^k(t). \tag{3.13}$$

It holds $\bigcup\{V_y;\ y \in G\} = G$. The space $\mathbf{R}^n$ is separable, therefore we can find a sequence $\{y_j\}$ of elements of G such that, we have $\bigcup_{j=1}^{\infty} V_{y_j} = G$. The measure H^k restricted to $\varphi(G)$ is σ-finite. Using Lemma 2.5 we find Borel sets $B, N \subset \varphi(G)$ such that $B \subset L \subset B \cup N$ and $H^k(N) = 0$. Using local lipschitzness of φ^{-1} we get $\lambda^k(\varphi^{-1}(N)) = H^k(\varphi^{-1}(N)) = 0$. Thus we obtain that the set $\varphi^{-1}(L)$ is λ^k-measurable. Set

$$A_j = \varphi^{-1}(L) \cap \Big(V_{y_j} \setminus \bigcup_{i=1}^{j-1} V_{y_i}\Big).$$

Then we have

(a) the set A_j is λ^k-measurable for every $j \in \mathbf{N}$,

(b) $A_j \subset V_{y_j}$ for every $j \in \mathbf{N}$,

(c) $\forall j, j' \in \mathbf{N}, j \neq j' \colon A_j \cap A_{j'} = \emptyset$,

(d) $\bigcup_{j=1}^{\infty} A_j = \varphi^{-1}(L)$,

(e) for every $j \in \mathbf{N}$ we have

$$\alpha^{-1} \int_{A_j} \operatorname{vol} \varphi'(t) \,\mathrm{d}\,\lambda^k(t) \leq H^k\big(\varphi(A_j)\big) \leq \alpha \int_{A_j} \operatorname{vol} \varphi'(t) \,\mathrm{d}\,\lambda^k(t),$$

(f) for every $j \in N$ the set $\varphi(A_j)$ is H^k-measurable.

From (a) and (c)–(e) we get

$$\alpha^{-1} \int_{\varphi^{-1}(L)} \operatorname{vol} \varphi'(t) \,\mathrm{d}\,\lambda^k(t) \leq H^k\big(\varphi(\varphi^{-1}(L))\big) \leq \alpha \int_{\varphi^{-1}(L)} \operatorname{vol} \varphi'(t) \,\mathrm{d}\,\lambda^k(t).$$

Since α has been chosen arbitrarily, we get (3.12).

2. Suppose that f is a nonnegative simple λ^k-measurable function, i.e., $f = \sum_{j=1}^{p} c_j \chi_{L_j}$, where $L_j \subset \varphi(G)$ is H^k-measurable and $c_j \geq 0$, $j = 1, \ldots, p$. Then by (3.12) we have

$$\begin{aligned} \int_{\varphi(G)} f(x) \,\mathrm{d}\, H^k(x) &= \sum_{j=1}^{p} c_j H^k(L_j) = \sum_{j=1}^{p} c_j \int_{\varphi^{-1}(L_j)} \operatorname{vol} \varphi'(t) \,\mathrm{d}\,\lambda^k(t) \\ &= \sum_{j=1}^{p} c_j \int_G \chi_{L_j} \circ \varphi(t) \operatorname{vol} \varphi'(t) \,\mathrm{d}\,\lambda^k(t) \\ &= \int_G f \circ \varphi(t) \operatorname{vol} \varphi'(t) \,\mathrm{d}\,\lambda^k(t). \end{aligned} \tag{3.14}$$

3. Let f be a nonnegative H^k-measurable function. We find a nonnegative simple H^k-measurable functions $f_j\colon \varphi(G) \to \mathbf{R}$, $j \in \mathbf{N}$, such that $f_j \to f$ a $f_j \leq f_{j+1}$. Then by Levi theorem we get

$$\lim_{j\to\infty} \int_{\varphi(G)} f_j(x)\,\mathrm{d}\,H^k(x) = \int_{\varphi(G)} f(x)\,\mathrm{d}\,H^k(x),$$
$$\lim_{j\to\infty} \int_G f_j(\varphi(t))\operatorname{vol}\varphi'(t)\,\mathrm{d}\,\lambda^k(t) = \int_G f(\varphi'(t))\operatorname{vol}\varphi'(t)\,\mathrm{d}\,\lambda^k(t).$$

Since using the point 2 we have for every $j \in \mathbf{N}$ the equality

$$\int_{\varphi(G)} f_j(x)\,\mathrm{d}\,H^k(x) = \int_G f_j(\varphi(t))\operatorname{vol}\varphi'(t)\,\mathrm{d}\,\lambda^k(t),$$

we get

$$\int_{\varphi(G)} f(x)\,\mathrm{d}\,H^k(x) = \int_G f(\varphi(t))\operatorname{vol}\varphi'(t)\,\mathrm{d}\,\lambda^k(t).$$

4. Let f be a H^k-measurable function and the integral $\int_G f(\varphi(t))\operatorname{vol}\varphi'(t)\,\mathrm{d}\,\lambda^k(t)$ converges. Set $f^+ = \max\{f, 0\}$ a $f^- = \max\{-f, 0\}$. By the point 3 it holds

$$\int_{\varphi(G)} f^+(x)\,\mathrm{d}\,H^k(x) = \int_G f^+(\varphi(t))\operatorname{vol}\varphi'(t)\,\mathrm{d}\,\lambda^k(t). \tag{3.15}$$

The last integral equals $\int_G (f(\varphi(t))\operatorname{vol}\varphi'(t))^+\,\mathrm{d}\,\lambda^k(t)$, thus it is finite by assumption. Similarly we get

$$\int_{\varphi(G)} f^-(x)\,\mathrm{d}\,H^k(x) = \int_G (f(\varphi(t))\operatorname{vol}\varphi'(t))^-\,\mathrm{d}\,\lambda^k(t), \tag{3.16}$$

the last integral is finite again. This implies

$$\int_{\varphi(G)} f(x)\,\mathrm{d}\,H^k(x) = \int_G f(\varphi(t))\operatorname{vol}\varphi'(t)\,\mathrm{d}\,\lambda^k(t).$$

□

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Remark. Area formula holds even for locally Lipschitz φ (cf. [1, F.34]).

Example. Compute $H^2(\mathbb{S}_2)$, where $\mathbb{S}_2 = \{x \in \mathbf{R}^3;\ \|x\| = 1\}$.

The set $\mathbb{S}_2$ can be written as a disjoint union $\mathbb{S}_2 = A_1 \cup A_2 \cup A_3$, where

$$\begin{aligned} A_1 &= \big\{[0,0,1], [0,0,-1]\big\}, \\ A_2 &= \{x \in \mathbb{S}_2;\ x_2 = 0, x_1 < 0\}, \\ A_3 &= \mathbb{S}_2 \setminus (A_1 \cup A_2). \end{aligned}$$

Using area formula we compute $H^2(A_3)$. We use spherical coordinate system $\varphi\colon G \to \mathbf{R}^3$, where $G = (-\pi,\pi) \times (-\pi/2, \pi/2)$ a

$$\varphi(\alpha,\gamma) = [\cos(\gamma)\cos(\alpha), \cos(\gamma)\sin(\alpha), \sin(\gamma)].$$

The mapping φ is injective regular and it holds $\varphi(G) = A_3$. We infer $\operatorname{vol}\varphi'(\alpha,\gamma) = \cos\gamma$ for $(\alpha,\gamma) \in G$. Then we have

$$\begin{aligned} H^2\big(\varphi(G)\big) &= \int_{\varphi(G)} 1 \,\mathrm{d}\, H^2 = \int_G \operatorname{vol}\varphi' \,\mathrm{d}\,\lambda^2 \\ &= \int_{-\pi}^{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos\gamma \,\mathrm{d}\,\gamma \,\mathrm{d}\,\alpha = 2\pi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos\gamma \,\mathrm{d}\,\gamma = 4\pi. \end{aligned}$$

It remains to show $H^2(A_1 \cup A_2) = 0$. The set A_1 has just two elements, thus we have $H^2(A_1) = 0$. The set A_2 can be parameterized by the mapping $\psi\colon (-\frac{\pi}{2}, \frac{\pi}{2}) \to \mathbf{R}^3$, which is defined by $\psi(t) = [-\cos t, 0, \sin t]$. The mapping ψ is injective regular and $\psi(-\frac{\pi}{2}, \frac{\pi}{2}) = A_2$. By area formula we obtain

$$\begin{aligned} H^1\big(\psi(-\tfrac{\pi}{2}, \tfrac{\pi}{2})\big) &= \int_{\psi(-\frac{\pi}{2}, \frac{\pi}{2})} 1 \,\mathrm{d}\, H^1 = \int_{(-\frac{\pi}{2}, \frac{\pi}{2})} \operatorname{vol}\psi' \,\mathrm{d}\,\lambda^1 \\ &= \int_{(-\frac{\pi}{2}, \frac{\pi}{2})} 1 \,\mathrm{d}\, t = \pi. \end{aligned}$$

By Theorem 3.1 we get $H^2(A_2) = 0$. We may conclude $H^2(\mathbb{S}_2) = 4\pi$.

Theorem 3.6 (coarea formula)**.** *Let $k, n \in \mathbf{N}, k > n$, $\varphi\colon \mathbf{R}^k \to \mathbf{R}^n$ be Lipschitz mapping, $f\colon \mathbf{R}^k \to \mathbf{R}$ be λ^k-integrable function. Then we have*

$$\begin{aligned} &\int_{\mathbf{R}^k} f(x)\sqrt{\det(\varphi'(x)\varphi'(x)^T)} \,\mathrm{d}\,\lambda^k(x) \\ &\qquad = \int_{\mathbf{R}^n} \Big(\int_{\varphi^{-1}(\{y\})} f(x) \,\mathrm{d}\, H^{k-n}(x)\Big) \,\mathrm{d}\,\lambda^n(y). \end{aligned}$$

Without proof.

Theorem 3.7. *Let $f\colon \mathbf{R}^k \to \mathbf{R}$ be λ^k-integrable function. Then we have*

$$\int_{\mathbf{R}^k} f(x) \,\mathrm{d}\,\lambda^k(x) = \int_0^\infty \Big(\int_{\{z\in\mathbf{R}^k;\ \|z\|=r\}} f(x) \,\mathrm{d}\, H^{k-1}(x)\Big) \,\mathrm{d}\,\lambda^1(r).$$

Chapter 4

Semicontinuous functions

Definition. let X be a topological space and $f\colon X \to \mathbf{R}^*$. We say that f is **lower semicontinuous**, if the set $\{x \in X;\ f(x) > a\}$ is open for every $a \in \mathbf{R}$. We say that f is **upper semicontinuous**, if the set $\{x \in X;\ f(x) < a\}$ is open for every $a \in \mathbf{R}$.

Notation. The abbreviations **lsc** and **usc** are used.

Theorem 4.1. *Let X be a metrizable topological space and $f\colon X \to \mathbf{R}^*$ be bounded from below. Then the function f is lsc, if and only if there exists a nondecreasing sequence $\{f_n\}$ of continuous functions from X to $\mathbf{R}$ such that $f_n \to f$.*

Chapter 5

Functions of Baire class 1

Definition. Let X and Y be metrizable topological spaces. A function $f\colon X \to Y$ is of **Baire class 1** (**B_1-function**) if for every open set U the set $f^{-1}(U)$ is F_σ.

Theorem 5.1 (Lebesgue–Hausdorff–Banach). *Let X be a metrizable topological space and $f\colon X \to \mathbf{R}$ be a B_1-function. Then there exists a sequence $\{f_n\}$ of continuous functions from X to $\mathbf{R}$ with $f_n \to f$.*

Lemma 5.2. *Let X be a metrizable topological space and $A \subset X$ be G_δ and F_σ set. Then χ_A is a pointwise limit of a sequence of continuous functions.*

The end of the lecture no. 3, 27. 2. 2023

Lemma 5.3. *Let X be a metrizable topological space, $p_n\colon X \to \mathbf{R}$, $n \in \omega$, be a pointwise limit of continuous functions. If the sequence $\{p_n\}$ converges uniformly to p, then p is a pointwise limit of continuous functions.*

Lemma 5.4 (reduction for F_σ sets). *Let X be a metrizable topological space, A_n be F_σ set for every $n \in \omega$. Then there are F_σ sets $A_n^* \subset A_n$, $n \in \omega$, such that $A_n^* \cap A_m^* = \emptyset$, whenever $n, m \in \omega$, $n \neq m$, and $\bigcup_{n\in\omega} A_n^* = \bigcup_{n\in\omega} A_n$.*

Remark. Theorem 5.1 holds also for X zero-dimensional and Y separable metrizable.

Theorem 5.5 (Baire). *Let X, Y be metrizable topological spaces, Y be separable, and $f\colon X \to Y$ be B_1-function. Then the set of points of continuity of f is residual and G_δ.*

The end of the lecture no. 4, 6. 3. 2023

Lemma 5.6. *Let X be a Polish topological space, i.e., separable topological space metrizable by a complete metric, $A, B \subset X$, $A \cap B = \emptyset$. If there is no set C which is G_δ and F_σ with $A \subset C$ and $C \cap B = \emptyset$, then there exists closed nonempty set F such that $A \cap F$, $B \cap F$ are dense in F.*

Proof. We define $F_0 = X$, $F_{\alpha+1} = \overline{A \cap F_\alpha} \cap \overline{B \cap F_\alpha}$, whenever $\alpha < \omega_1$, and $F_\eta = \bigcap_{\alpha<\eta} F_\alpha$, whenever $\eta < \omega_1$ is a limit ordinal. Then $(F_\alpha)_{\alpha<\omega_1}$ is a nonincreasing sequence of closed sets in X. One can infer that there exists $\zeta < \omega_1$ such that $F_\zeta = F_{\zeta+1}$.

Claim. $F_\zeta \neq \emptyset$

Proof of Claim. We assume towards contradiction that $F_\zeta = \emptyset$. Then we can write

$$X = \bigcup_{\alpha<\zeta} (F_\alpha \setminus F_{\alpha+1}). \tag{5.1}$$

We set $C = \bigcup_{\alpha<\zeta} \big(\overline{A \cap F_\alpha} \setminus F_{\alpha+1}\big)$. Then one can get $A \subset C$ and $C \cap B = \emptyset$. We have that C is F_σ as well as G_δ. To check the latter fact we define G_δ sets

$$G_\alpha = \overline{A \cap F_\alpha} \cup (X \setminus F_\alpha) \cup F_{\alpha+1}, \qquad \alpha < \zeta,$$

and we verify that

$$C = \bigcap_{\alpha<\zeta} G_\alpha.$$

The inclusion $\subset$. For $x \in C$ there exists $\alpha_0 < \omega_1$ such that $x \in \overline{A \cap F_{\alpha_0}} \setminus F_{\alpha_0+1}$. Take $\alpha < \omega_1$. We distinguish the following three possibilities. If $\alpha < \alpha_0$, then

$$x \in \overline{A \cap F_{\alpha_0}} \subset F_{\alpha_0} \subset F_{\alpha+1} \subset G_\alpha.$$

If $\alpha = \alpha_0$, then

$$x \in \overline{A \cap F_{\alpha_0}} \subset G_{\alpha_0} = G_\alpha.$$

If $\alpha > \alpha_0$ then

$$x \in X \setminus F_{\alpha_0+1} \subset X \setminus F_\alpha \subset G_\alpha.$$

The inclusion $\supset$. Now suppose that $x \in \bigcap_{\alpha<\zeta} G_\alpha$. By (5.1) there exists $\beta < \zeta$ with $x \in F_\beta \setminus F_{\beta+1}$. We also have $x \in G_\beta$. This implies that $x \in \overline{A \cap F_\beta} \setminus F_{\beta+1} \subset C$.

Thus C is a G_δ and F_σ set separating A form B, a contradiction. This finishes the proof of Claim. □

Now it is sufficient to set $F = F_\zeta$. □

Theorem 5.7 (Baire)**.** *Let X be Polish, Y separable metrizable, and $f\colon X \to Y$. Then the following are equivalent*

(i) *f is a B_1-function.*

(ii) *$f|_F$ has a point of continuity for every $F \subset X$ closed.*

Chapter 6

Density topology, approximate continuity and differentiability

Definition. Let f be a function from $\mathbf{R}$ to $\mathbf{R}$, $a \in \mathbf{R}$, and $L \in \mathbf{R}$. We say that f has **approximate limit** L at the point a if

$$\forall \varepsilon > 0\ \exists \delta > 0\ \forall B \in \mathcal{B}, a \in B, \operatorname{diam} B < \delta\colon\ \lambda_n^*\bigl(\{x \in B;\ |f(x) - L| \geq \varepsilon\}\bigr) < \varepsilon \lambda_n(B).$$

Theorem 6.1. *Let f be a function from $\mathbf{R}$ to $\mathbf{R}$, $a \in \mathbf{R}$. Then f has at most one approximate limit at a.*

Notation. Let f be a function from $\mathbf{R}$ to $\mathbf{R}$. The approximate limit of f at $a \in \mathbf{R}$ is denoted by ap-$\lim_{x \to a} f(x)$.

Definition. A function from $\mathbf{R}$ to $\mathbf{R}$ is **approximately continuous** at $a \in \mathbf{R}$ if ap-$\lim_{x \to a} f(x) = f(a)$.

——— The end of the lecture no. 5, 20. 3. 2023 ———

Definition. We say that a measurable set $A \subset \mathbf{R}$ is d**-open**, if each point of A is a point of density of A.

Theorem 6.2. *The system of d-open sets forms a topology.*

Notation. The symbol τ_d stands for the **density topology** from the previous theorem.

Properties of density topology

- The topology τ_d is finer than the standard topology.
- The topology τ_d is not metrizable.
- A set $K \subset \mathbf{R}$ is τ_d-compact if and only if K is finite.

- The topology τ_d is not normal.
- Baire theorem holds in $(\mathbf{R}, \tau_d)$.

Theorem 6.3. *The topology τ_d is completely regular, i.e., if $F \subset \mathbf{R}$ is τ_d-closed and $x_0 \in \mathbf{R} \setminus F$, then there exists τ_d-continuous function $f\colon \mathbf{R} \to [0,1]$ such that $f(y) = 0$ for every $y \in F$ and $f(x_0) = 1$.*

Lemma 6.4. *Let $E \subset \mathbf{R}$ be measurable, $X \subset E$ is closed and $\mathrm{d}(E, x) = 1$ for every $x \in X$. Then there exists a closed set $P \subset \mathbf{R}$ such that*

- *$X \subset P \subset E$,*
- *$\forall x \in X\colon\ \mathrm{d}(P, x) = 1$,*
- *$\forall x \in P\colon\ \mathrm{d}(E, x) = 1$.*

The end of the lecture no. 6, video lecture

Remark. Let f be a function from $\mathbf{R}$ to $\mathbf{R}$.

(a) The function f is approximately continuous at $a \in \mathbf{R}$ if and only if f is τ_d-continuous at a.

(b) The function f is approximately continuous at $a \in \mathbf{R}$ if and only there exists a measurable set $M \subset \mathbf{R}$ such that $\mathrm{d}(M, a) = 1$ and $\lim_{x \to a, x \in M} f(x) = f(a)$.

Theorem 6.5 (Denjoy)**.** *Let $f\colon \mathbf{R} \to \mathbf{R}$. Then the function f is approximately continuous a.e. if and only if f is measurable.*

Proof. $\Rightarrow$ We set

$$N = \{x \in \mathbf{R};\ f \text{ is not approximately continuous at } x\}.$$

Then we have $\lambda_1(N) = 0$. Choose $c \in \mathbf{R}$ and set $M = \{x \in \mathbf{R};\ f(x) > c\}$. The set $M \setminus N$ is d-open, therefore it is a measurable set. This implies that M is measurable. Consequently, we have that f is measurable.

$\Leftarrow$ Choose $\varepsilon > 0$. By Luzin theorem there exist a closed set $F \subset \mathbf{R}$ with $\lambda_1(\mathbf{R} \setminus F) < \varepsilon$ and a function $g\colon F \to \mathbf{R}$ which is continuous on F satisfying $f|_F = g$. We have that a.e. point in F is a density point of F, therefore f is approximately continuous at a.e. point in F. This implies that f is approximately continuous a.e. in $\mathbf{R}$. $\square$

Theorem 6.6. *Let $f\colon \mathbf{R} \to \mathbf{R}$ be a bounded approximately continuous function. Then f has an antiderivative on $\mathbf{R}$.*

Proof. Find $K \in \mathbf{R}$ such that $|f(x)| \leq K$ for every $x \in \mathbf{R}$. We set $F(x) = \int_0^x f$. The function f is measurable by Theorem 6.5 and is bounded, therefore F is well defined. Choose $x \in \mathbf{R}$. Let $\varepsilon > 0$. We find $\delta > 0$ such that for every $h \in (0, \delta)$ it holds

$$\frac{1}{h}\lambda_1\big(\{y \in [x, x+h];\ |f(y) - f(x)| \geq \varepsilon\}\big) < \varepsilon.$$

Fix $h \in (0, \delta)$ and denote $M = \{y \in [x, x+h];\ |f(y) - f(x)| \geq \varepsilon\}$. It holds

$$\begin{aligned}
\Big|\frac{1}{h}\big(F(x+h) - F(x)\big) - f(x)\Big| &= \frac{1}{h}\Big|\int_x^{x+h} (f(t) - f(x))\, dt\Big| \\
&\leq \frac{1}{h}\int_M |f(t) - f(x)|\, dt + \frac{1}{h}\int_{[x,x+h]\setminus M} |f(t) - f(x)|\, dt \\
&\leq \frac{1}{h}2K \cdot \varepsilon h + \frac{1}{h} \cdot h\varepsilon = (2K + 1)\varepsilon.
\end{aligned}$$

This implies $F'_+(x) = f(x)$. One can infer $F'_-(x) = f(x)$ analogously. $\square$

The end of the lecture no. 7, 27. 3. 2023

Corollary 6.7. *Let $f \colon \mathbf{R} \to \mathbf{R}$ be a bounded approximately continuous function. Then f has Darboux property and is in B_1.*

Theorem 6.8. *There exists a differentiable function $f \colon \mathbf{R} \to \mathbf{R}$ such that the sets $\{x \in \mathbf{R};\ f'(x) > 0\}$ and $\{x \in \mathbf{R};\ f'(x) < 0\}$ are dense.*

Proof. Let $A, B \subset \mathbf{R}$ be countable, dense, and disjoint. Suppose that $A = \{a_n;\ n \in \mathbf{N}\}$ and $B = \{b_n;\ n \in \mathbf{N}\}$. Observe that A and B are τ_d-closed. Using Theorem 6.3 we find for every $n \in \mathbf{N}$ approximately continuous functions g_n and h_n such that

$$\begin{aligned}
g_n(a_n) &= 1, & h_n(b_n) &= 1, \\
0 \leq g_n &\leq 1, & 0 \leq h_n &\leq 1, \\
g_n|_B &= 0, & h_n|A &= 0.
\end{aligned}$$

We define

$$\psi = \sum_{n=1}^{\infty} 2^{-n} g_n - \sum_{n=1}^{\infty} 2^{-n} h_n.$$

Then the function ψ is bounded, approximately continuous, positive on A, and negative on B. By Theorem 6.6 there is a function f such that $f' = \psi$ and we are done. $\square$

Remark. We say that a differentiable function g is of **Köpcke type** if g' is bounded and the sets $\{g' > 0\}$ and $\{g' < 0\}$ are dense.

Chapter 7

More on derivatives

Notation. Let I be a nonempty open interval. The set of all real functions defined on I which have an antiderivative on I is denoted by $\Delta'(I)$.

Remark. We have ap $- \mathcal{C}_b(I) \subset \Delta'(I) \subset \mathcal{D}B_1(I)$.

Theorem 7.1. *Let I be a nonempty open interval and $f \in \Delta'(I)$ The f has Denjoy-Clarskon property, i.e., for every open $G \subset \mathbf{R}$ we have that either $f^{-1}(G) = \emptyset$ or $\lambda(f^{-1}(G)) > 0$.*

Proof. To be added. □

——— The end of the lecture no. 8, 3. 4. 2023 ———

Lemma 7.2. *Let F be a differentiable at each point of the interval $[a, b] \subset \mathbf{R}$ and F' is bounded from below. Then F is absolutely continuous on $[a, b]$.*

Proof. To be added. □

Theorem 7.3. *Let f be differentiable at each point of $[a, b] \subset \mathbf{R}$ and $f' \in L^1([a, b])$. Then we have*

$$f(x) - f(a) = (L) \int_a^x f'(t) \,\mathrm{d}\, t, \qquad x \in [a, b].$$

Proof. To be added. See [3, 7.21]. □

Theorem 7.4 (Caratheodory–Vitali)**.** *Let $f\colon \mathbf{R} \to \mathbf{R}$ satisfy $f \in L^1(\lambda)$ and $\varepsilon > 0$. Then there exists $u, v\colon \mathbf{R} \to \mathbf{R}^*$ such that*

- *$u \leq f \leq v$,*
- *u is usc and bounded from above,*
- *v is lsc and bounded from below,*
- *$\int (u - v) \,\mathrm{d}\, \lambda < \varepsilon$.*

Proof. To be added. See [3]. □

——— The end of the lecture no. 9, 17. 4. 2023 ———

Zahorski classes

Definition. Let $E \subset \mathbf{R}$ be an F_σ set. We say that E belongs to class

M_0 if every point of E is a point of bilateral accumulation of E,

M_1 if every point fo E is a point of bilateral condensation of E,

M_2 if each one sided neighbourhood of each $x \in E$ intersects E in a set of positive measure,

M_3 if for each $x \in E$ and each sequence $\{I_k\}$ of closed intervals converging to x such that $\lambda(I_n \cap E) = 0$ for each n, we have

$$\lim_{n\to\infty} \frac{\lambda(I_n)}{\operatorname{dist}(x, I_n)} = 0,$$

M_4 if there exists a sequence of closed sets $\{K_n\}$ and a sequence of positive numbers η_n such that $E = \bigcup_{n=1}^{\infty} K_n$ and

$$\forall n \in \mathbf{N}\ \forall x \in K_n \forall c > 0\ \exists \varepsilon > 0$$
$$\forall h, h_1 \in \mathbf{R}, hh_1 > 0, \frac{h}{h_1} < c, |h + h_1| < \varepsilon \colon \frac{\lambda(E \cap (x+h, x+h+h_1))}{|h_1|} > \eta_n.$$

M_5 if every point of E is a point of density of E.

TO BE COMPLETED

Chapter 8

Sets with a finite perimeter and divergence theorem

8.1 Motivation

Lemma 8.1. *Let F be distribution function of a signed Radon measure μ and $\varphi \in \mathcal{C}_c^1(\mathbf{R})$. Then $\int \varphi \,\mathrm{d}\,\mu = -\int F\varphi' \,\mathrm{d}\,\lambda$.*

Theorem 8.2. *Let $u \in L^1(\mathbf{R})$. Then the following are equivalent.*

(i) *There exists a signed Radon measure μ such that the weak derivative of u is μ.*

(ii) *There exists $v\colon \mathbf{R} \to \mathbf{R}$ such that $v \in BV([a,b])$ for every $a, b \in \mathbf{R}$, $a < b$, and $v = u$ a.e.*

Theorem 8.3 (Gauss divergence theorem)**.** *Let $n > 1, \Omega \subset \mathbf{R}^n$ be a bounded nonempty open set with $H^{n-1}(\partial\Omega) < \infty$, $H^{n-1}(\partial\Omega \setminus \partial_r\Omega) = 0, f \in \mathcal{C}^1(\overline{\Omega}, \mathbf{R}^n)$. Then we have*

$$\int_{\partial\Omega} \langle f(y), \nu_\Omega(y)\rangle \,\mathrm{d}\, H^{n-1}(y) = \int_\Omega \operatorname{div} f(x) \,\mathrm{d}\,\lambda^n(x).$$

Without proof.

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8.2 Definitions and remarks

Definition. Let $U \subset \mathbf{R}^n$ be open.

(a) A function $f \in L^1(U)$ has **bounded variation in** U if

$$\sup\Big\{\int_U f \operatorname{div} \varphi \,\mathrm{d}\, x;\ \varphi \in \mathcal{C}^1_c(U, \mathbf{R}^n), \|\varphi\|_\infty \le 1\Big\} < \infty.$$

We write $BV(U)$ to denote the space of functions of bounded variation.

(b) A λ^n-measurable set $E \subset \mathbf{R}^n$ has **finite perimeter in** U if $\chi_E \in BV(U)$.

(c) A function $f \in L^1(U)$ has **locally bounded variation in** U if for each open set V with $\overline{V} \subset U$ we have

$$\sup\Big\{\int_U f \operatorname{div} \varphi \,\mathrm{d}\, x;\ \varphi \in \mathcal{C}^1_c(V, \mathbf{R}^n), \|\varphi\|_\infty \le 1\Big\} < \infty.$$

We write $BV_{loc}(U)$ to denote the space of such functions.

(d) A λ^n-measurable set $E \subset \mathbf{R}^n$ has **locally finite perimeter in** U if $\chi_E \in BV_{loc}(U)$.

Theorem 8.4 (structure theorem)**.** *Let $U \subset \mathbf{R}^n$ be open and $f \in BV_{loc}(U)$. Then there exists a Radon measure μ on U and a μ-measurable function $\sigma\colon U \to \mathbf{R}^n$ such that*

(a) $\|\sigma(x)\| = 1$ *μ a.e. and*

(b) $\int_U f \operatorname{div} \varphi \,\mathrm{d}\, x = -\int_U \langle \varphi, \sigma\rangle \,\mathrm{d}\, \mu$ *for every $\varphi \in \mathcal{C}^1_c(U, \mathbf{R}^n)$.*

Remark. (1) Let $U \subset \mathbf{R}^n$ be open, $f \in BV_{loc}(U)$, $i \in \{1, \dots, n\}$. For $\psi \in \mathcal{C}^\infty_c(U)$ we set $\varphi = [0, \dots, 0, \psi, 0, \dots, 0]$ and we have

$$\int_U f\partial_i \psi = \int_U f \operatorname{div} \varphi = -\int_U \langle \varphi, \sigma\rangle \,\mathrm{d}\, \mu = -\int_U \psi \sigma_i \,\mathrm{d}\, \mu.$$

If $f \in L^1_{loc}(U)$ and $\partial_i f$, $i = 1, \dots, n$, is a signed Radon measure, then $f \in BV_{loc}(U)$. We have

$$\int_U f\partial_i \psi = -\int \psi \,\mathrm{d}\, \tau_i, \qquad \psi \in \mathcal{C}^\infty_c(U).$$

Then for every $\varphi \in \mathcal{C}^\infty_c(V, \mathbf{R}^n)$, where V is open with $\overline{V} \subset U$, we have

$$\int_U f \operatorname{div} \phi = -\int_U \sum_{i=1}^n \psi$$

Theorem 8.5 (lower semicontinuity of variation measure)**.** *Let $U \subset \mathbf{R}^n$ be open, $f_k \in BV(U)$, $k \in \mathbf{N}$, and $f_k \to f$ in $L^1_{loc}(U)$. Then*

$$\|Df\|(U) \le \liminf_{k\to\infty} \|Df_k\|(U).$$

8.3 Coarea formula for BV functions

Notation. For $f\colon U \to \mathbf{R}$ and $t \in \mathbf{R}$, define $E_t = \{x \in U; f(x) > t\}$.

Lemma 8.6. *Let $U \subset \mathbf{R}^n$ be open and $f \in BV(U)$. Then the mapping $t \mapsto \|\partial E_t\|(U)$ is λ^1-measurable.*

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Proof. The mapping $(x,t) \mapsto \chi_{E_t}(x)$ is λ^{n+1}-measurable since the set

$$\{(x,t) \in U \times \mathbf{R};\ \chi_{E_t}(x) = 1\} = \{(x,t) \in U \times \mathbf{R};\ x \in E_t\} = \{(x,t) \in U \times \mathbf{R};\ f(x) > t\}$$

is a subgraph of the measurable function f. Let $\varphi \in \mathcal{C}^1_c(U, \mathbf{R}^n)$. Then the function

$$t \mapsto \int_{E_t} \operatorname{div} \varphi = \int \chi_{E_t} \operatorname{div} \varphi$$

is λ^1-measurable. Find $D \subset \mathcal{C}^1_c(U, \mathbf{R}^n)$ countable and dense. Then

$$t \mapsto \|\partial E_t\|(U) = \sup_{\|\varphi\|_\infty \le 1, \varphi \in \mathcal{C}^1_c(U,\mathbf{R}^n)} \int_{E_t} \operatorname{div} \varphi = \sup_{\|\varphi\|_\infty \le 1, \varphi \in D} \int_{E_t} \operatorname{div} \varphi$$

is λ^1-measurable.

□

Theorem 8.7. *Let $U \subset \mathbf{R}^n$ be open and $f \in BV(U)$. Then*

- *E_t has finite perimeter for λ^1-a.e. $t \in \mathbf{R}$ and*
- *$\|Df\| = \int_{-\infty}^{\infty} \|\partial E_t\|(U)\,\mathrm{d}\,t$.*
- *Conversely, if $f \in L^1(U)$ and $\int_{-\infty}^{\infty} \|\partial E_t\|(U)\,\mathrm{d}\,t < \infty$, then $f \in BV(U)$.*

8.4 The reduced boundary

To be added.

8.5 Gauss theorem for BV functions

To be added.

The end of the lecture no. 12, 22. 5. 2023

The end of Summer Semester

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