Synthetic projective methods used for solving problems in geometry

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K. G. Ch. von Staudt

Karl Georg Christian von Staudt

1798-1867

German mathematician

student of Gauss

1835 professor at Erlangen

Selected works:

1847	Geometrie der Lage
1856 - 60	Beiträge zur Geometrie der Lage

K. G. Ch. von Staudt

". . . Yet man is no mole. Infinite feelers radiate from the windows of his soul, whose wings touch the fixed stars. The angel of light in him created for the guidance of eye-life an independent system, a radiant geometry, a visual space, codified in 1847 by a new Euclid, by the Erlangen professor, Georg von Staudt, in his immortal Geometrie der Lage published in the quaint and ancient Nürnberg of Albrecht Dürer. . . "

The preface of *Synthetic Projective Geometry*, by G. B. Halsted

K. G. Ch. von Staudt

Von Staudt's contribution in projective geometry:

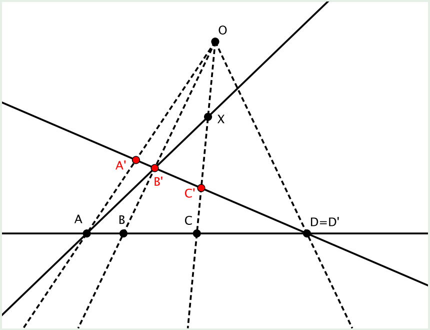
- projective geometry (geometry of position) derived synthetically - "Wurf"
- synthetic construction of the algebraic structure of field
- the fundamental theorem of projective geometry (in $\mathbb{R}$)
- synthetic construction of imaginary elements

Method of projection

- Central perspectivity preserves Wurf
- Any perspectivity or a composition of two or more perspectivities is a projectivity

Example

Wurfs of collinear points *ABCD* and *BADC* are equal.



ABCD

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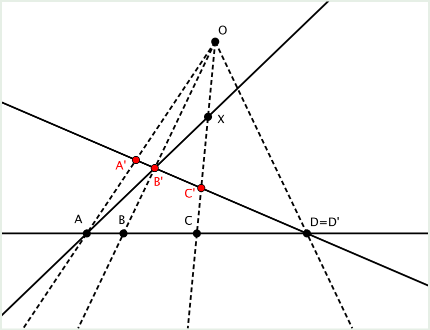

$$ABCD \overset{(O)}{\overline{\wedge}} A'B'C'D$$

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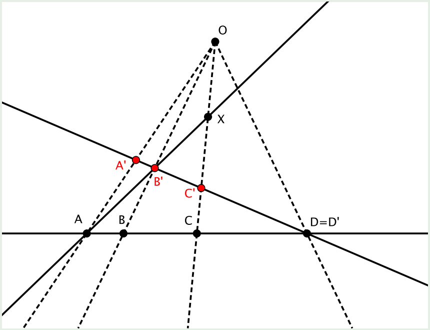

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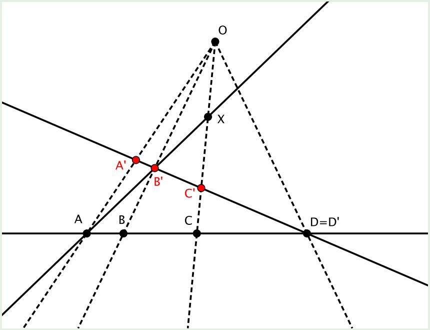

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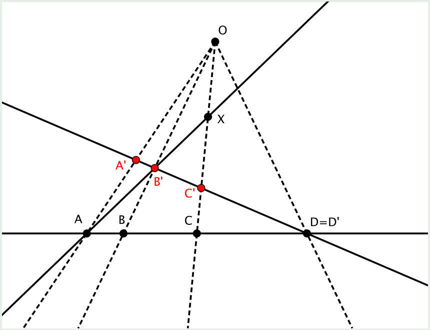

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$$ABCD \,\overline{\wedge}\, BADC$$

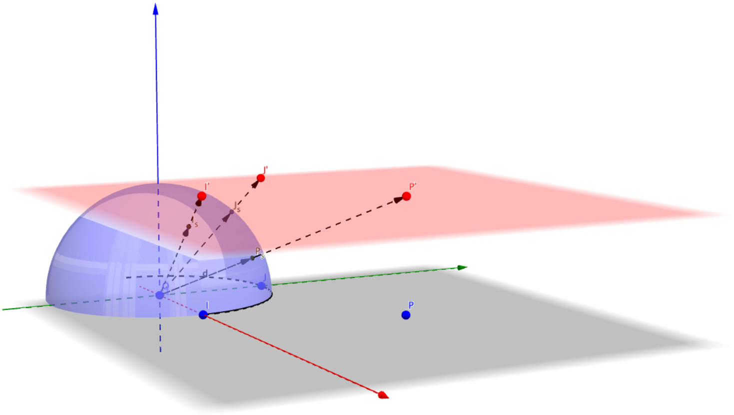
Projective space

Definitions of the (real) projective space:

- projective extension of the real space
- axiomatically - built on the incidence property
- existentially - as a real vector space

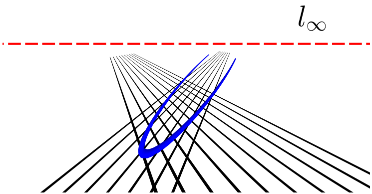
affine plane → projective plane

afiine points $[x, y] \to$ projective points (x_1, x_2, x_0); $x_0 \neq 0$
affine directions (infinite points) $(x, y) \to$ projective points $(x_1, x_2, 0)$

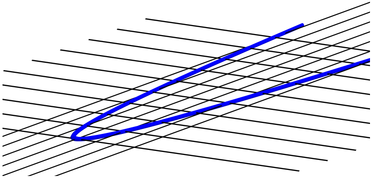


Substracting one line l_∞ from $\mathbb{P}^2$

projective



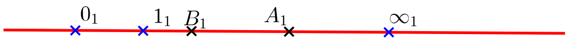
affine



von Staudt's constructions

A choice of *the projective coordinate system* of an one-dimensional form - elements $0, 1, \infty$.

The sum of elements *A*, *B* (Wurfs $01\infty A$; $01\infty B$)



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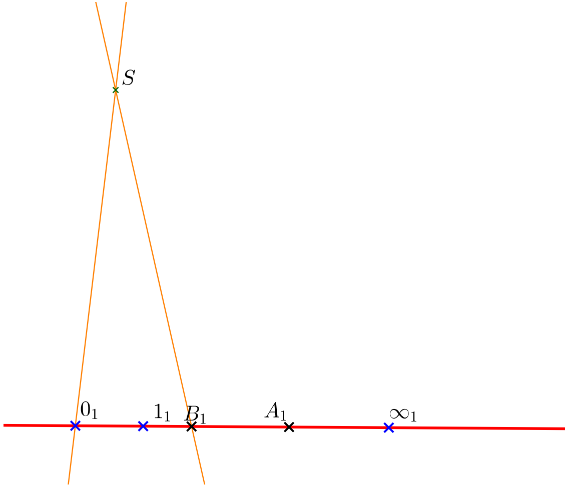

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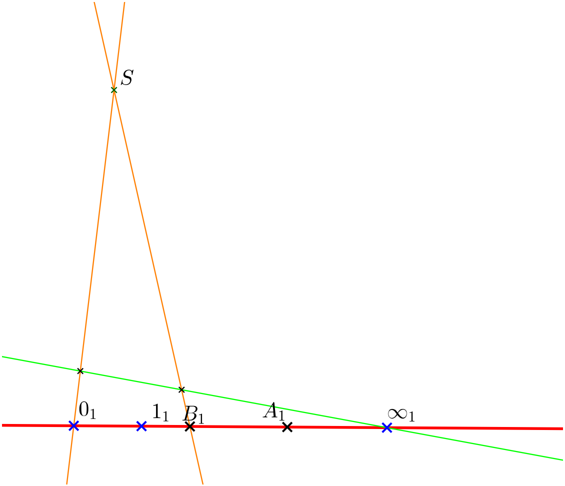

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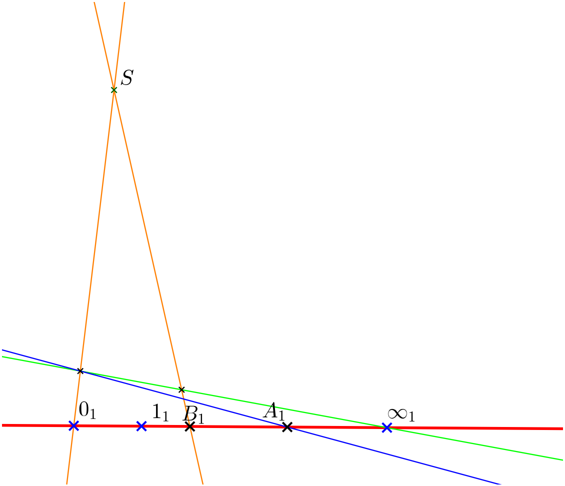

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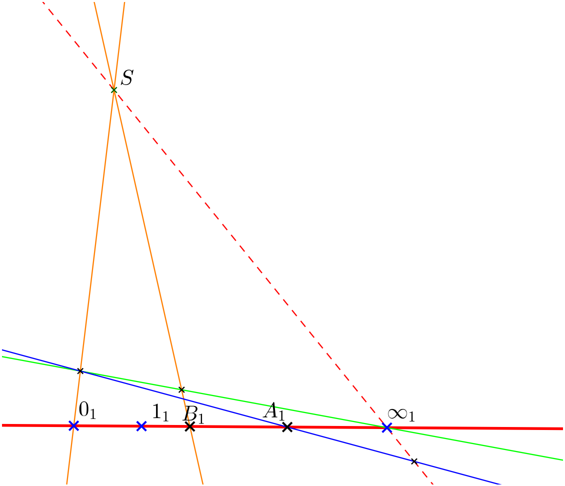

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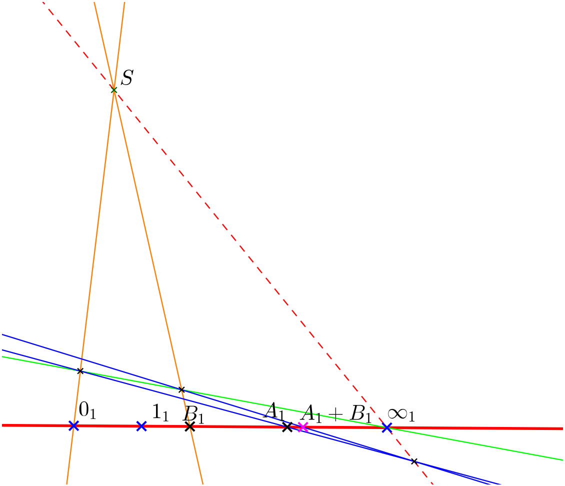

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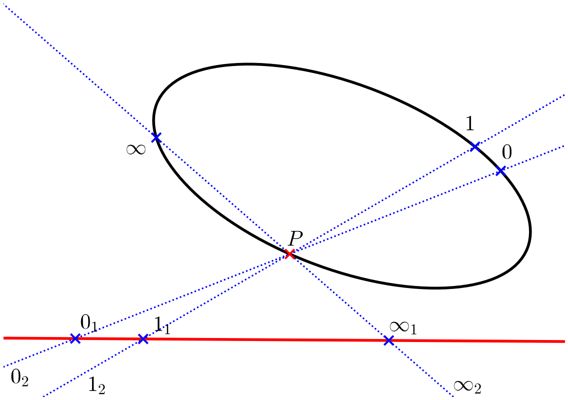

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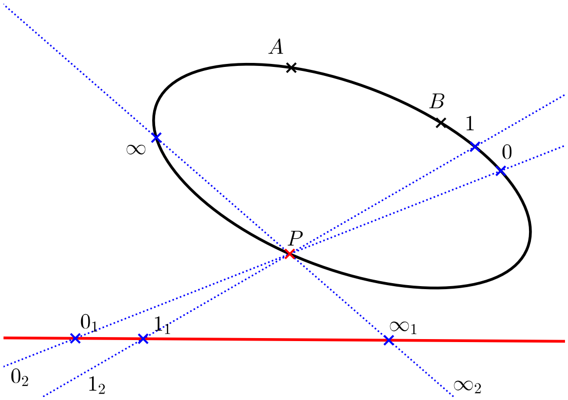

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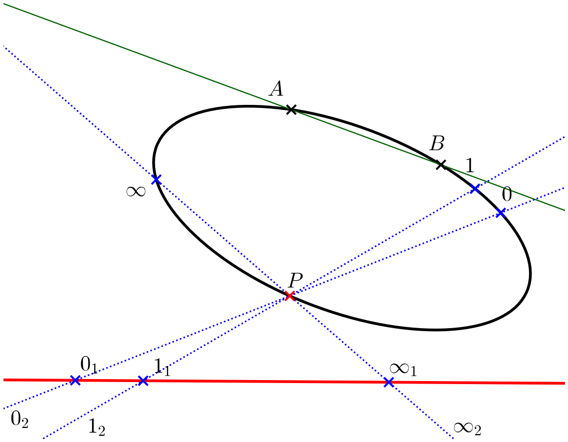

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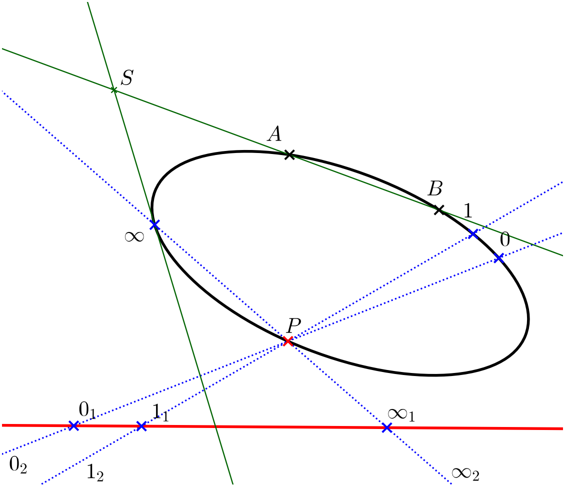

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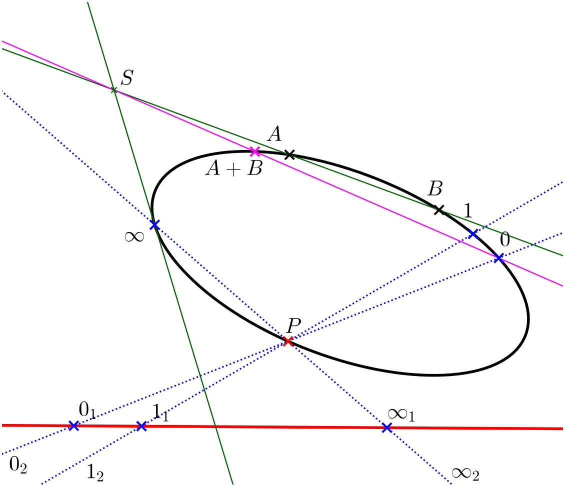

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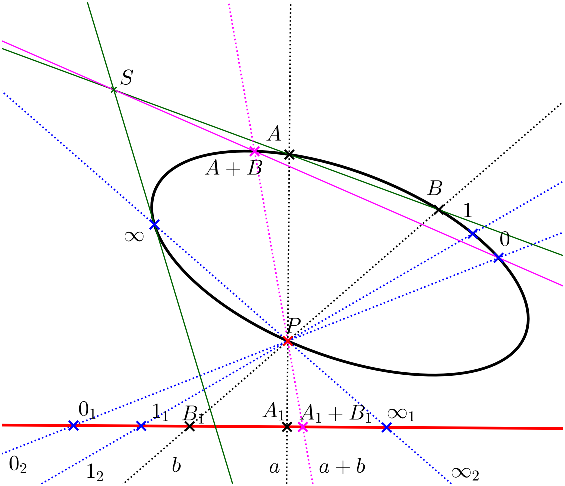
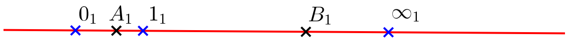


von Staudt's constructions

The product of elements A, B (Wurfs $01\infty A$; $01\infty B$)



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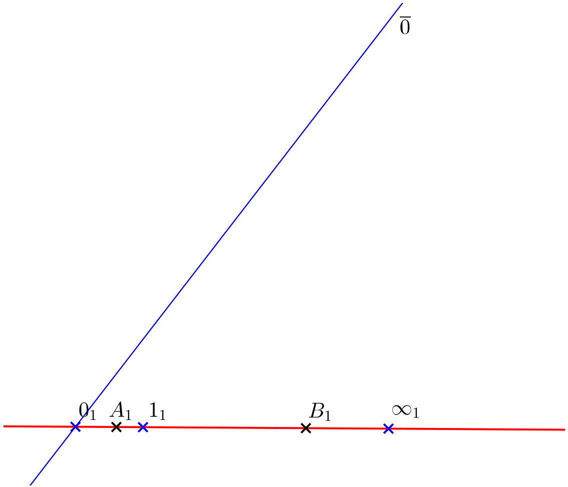

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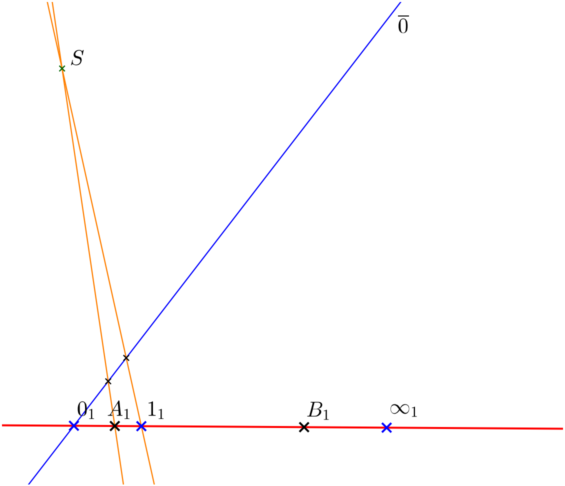

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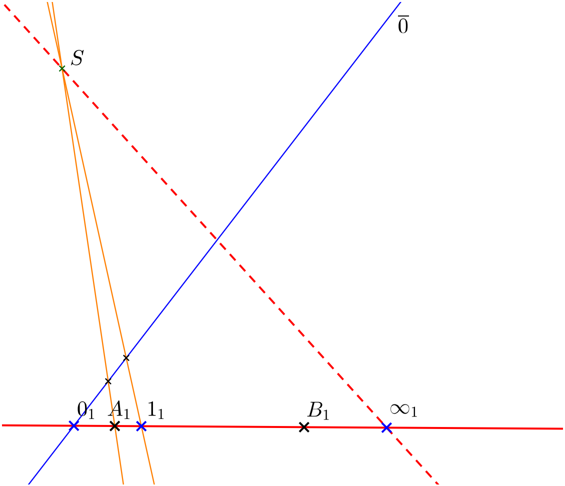

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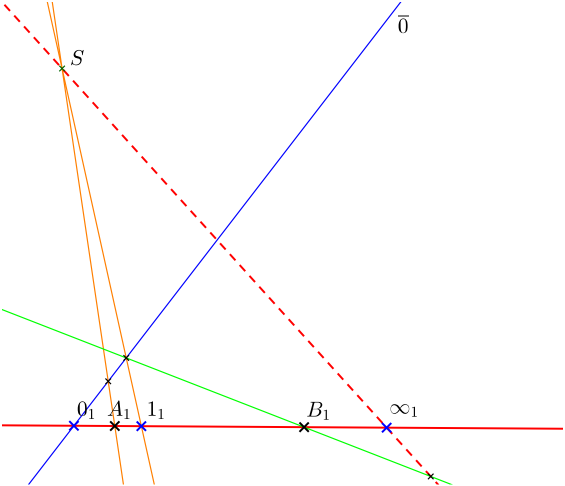

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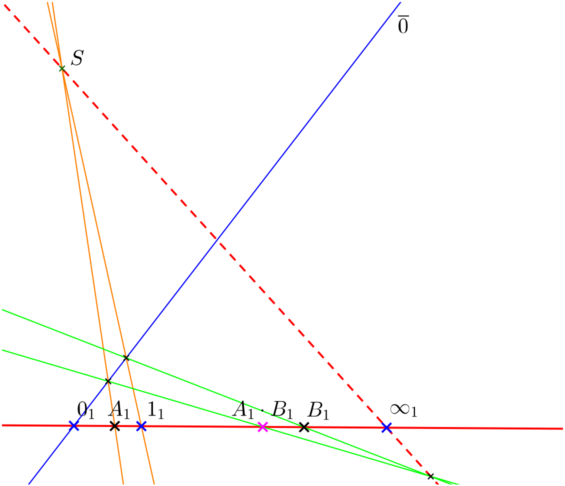

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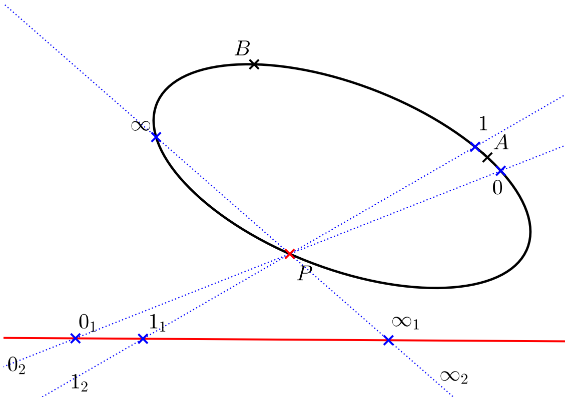

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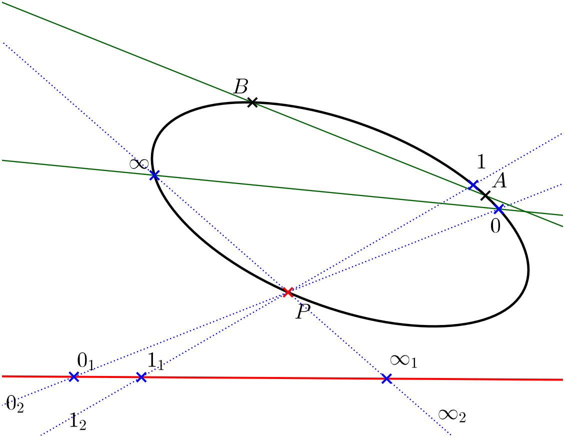

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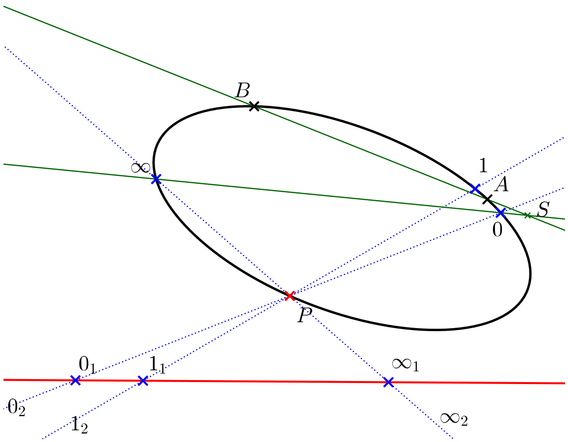

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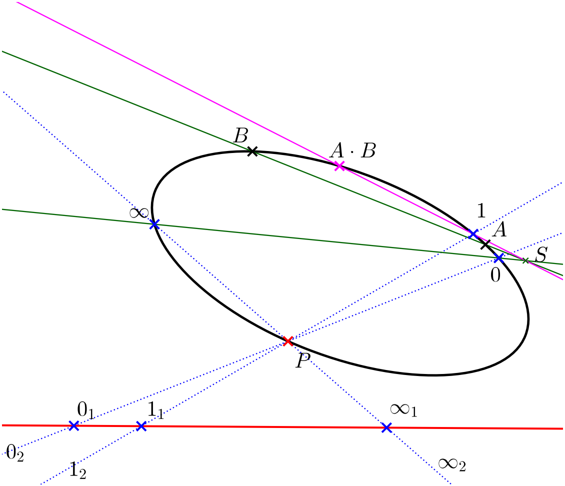

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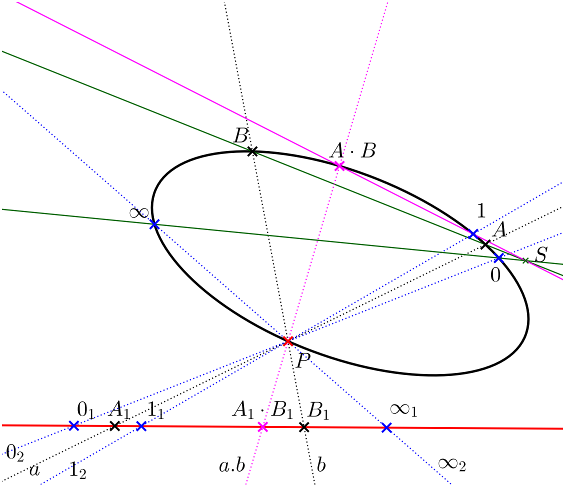

Cross-ratio

Definition (Cross-ratio in projective coordinates)

Let $a(a_1, a_0), b(b_1, b_0)$, $c(c_1, c_0)$, $d(d_1, d_0)$ be four elements of the same one-dimensional form. Then

$$(a, b; c, d) = \frac{\begin{vmatrix} a_1 & a_0 \\ c_1 & c_0 \end{vmatrix} \begin{vmatrix} b_1 & b_0 \\ d_1 & d_0 \end{vmatrix}}{\begin{vmatrix} b_1 & b_0 \\ c_1 & c_0 \end{vmatrix} \begin{vmatrix} a_1 & a_0 \\ d_1 & d_0 \end{vmatrix}} = \frac{(a_1 c_0 - a_0 c_1)(b_1 d_0 - b_0 d_1)}{(b_1 c_0 - b_0 c_1)(a_1 d_0 - a_0 d_1)}$$

is said to be the cross ratio of elements a, b, c, d (in this order).

Cross-ratio in homogenous coordinates

Let $A(a_1, a_0)$, $B(b_1, b_0)$, $C(c_1, c_0)$, $D(d_1, d_0)$ be four points in $\mathbb{RP}^1$. Then

$$(A, B; C, D) = \frac{\begin{vmatrix} a_1 & a_0 \\ c_1 & c_0 \end{vmatrix} \begin{vmatrix} b_1 & b_0 \\ d_1 & d_0 \end{vmatrix}}{\begin{vmatrix} b_1 & b_0 \\ c_1 & c_0 \end{vmatrix} \begin{vmatrix} a_1 & a_0 \\ d_1 & d_0 \end{vmatrix}}$$

Let $A(a_1, a_2, a_0)$, $B(b_1, b_2, b_0)$, $C(c_1, c_2, c_0)$, $D(d_1, d_2, d_0)$ be four points on a line l in $\mathbb{RP}^2$ and let $O(o_1, o_2, o_0)$ be a point not on l. Then

$$(A, B; C, D) = \frac{[OAC] \cdot [OBD]}{[OBC] \cdot [OAD]}$$

D is the *harmonic conjugate* of C with respect to A and B.
$(A, B; C, D) = -1$

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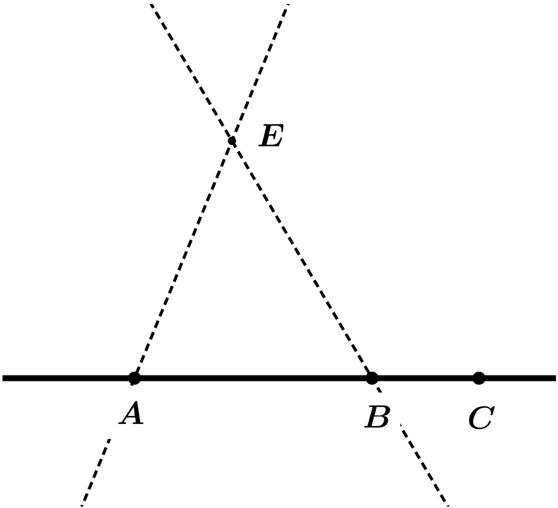

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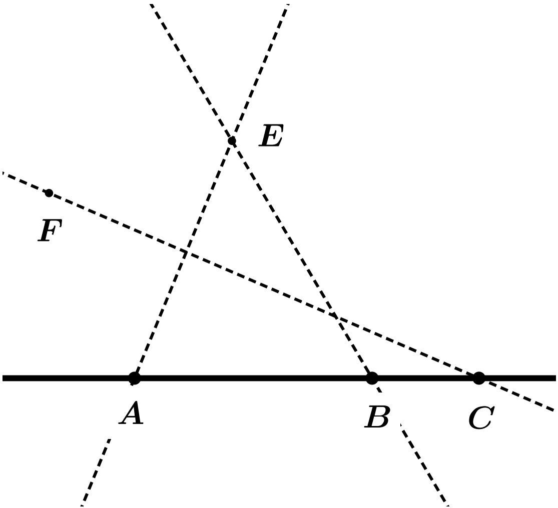

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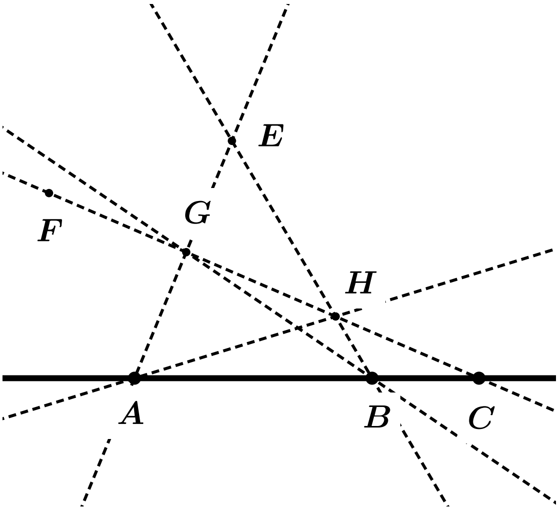

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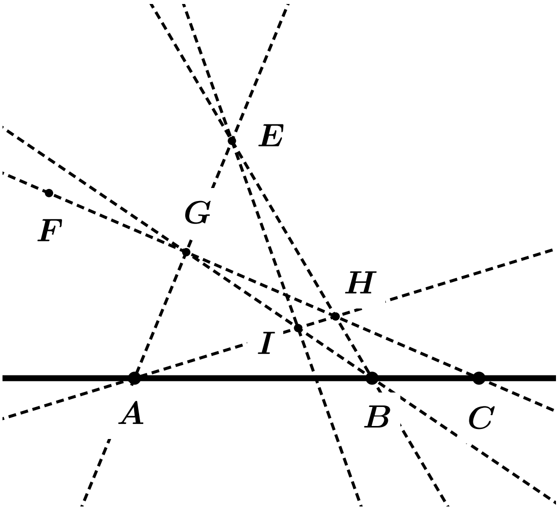

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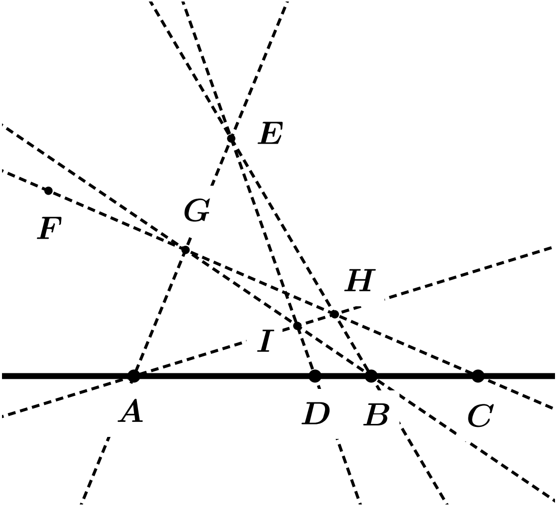

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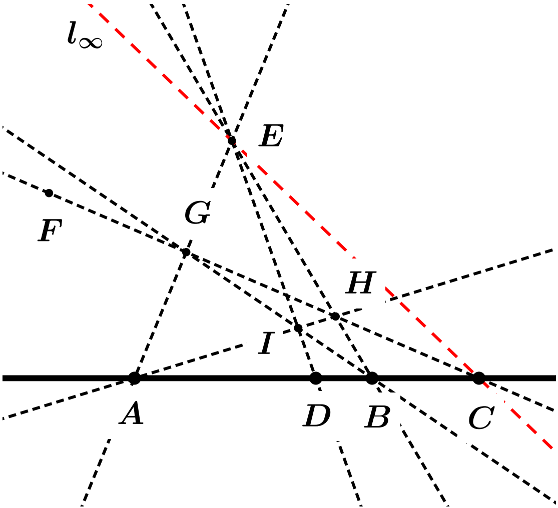

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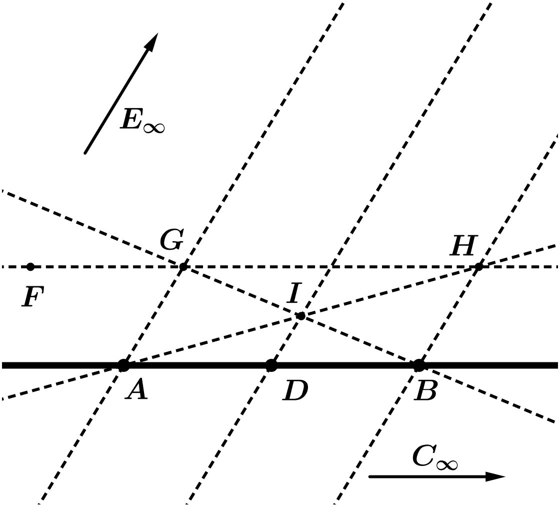

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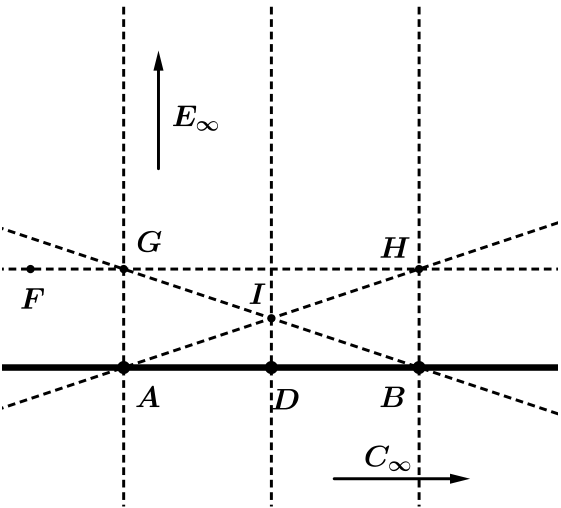
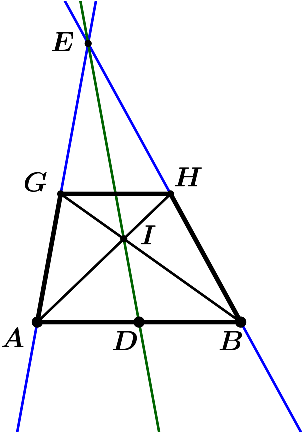


One elementary theorem

Let *ABHG* be a trapezoid, where $AB \| HG$ and *E* be the intersection point of *AG* and *BF*. The intersection point *I* of diagonals *AH* and *BG* lies on the median of the triangle *ABE* through *E*.



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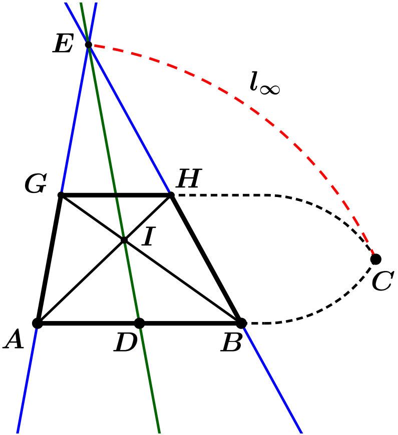
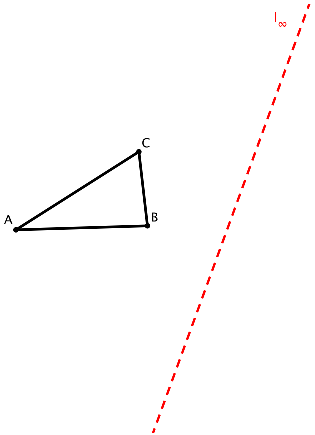
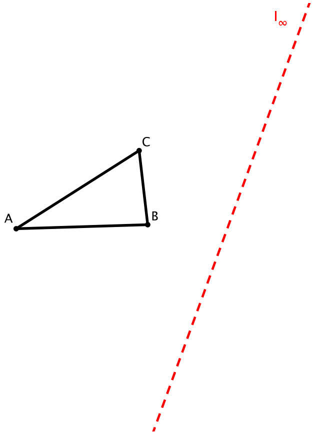


A centroid of a triangle (with respect to a line)



With a little help from GeoGebra



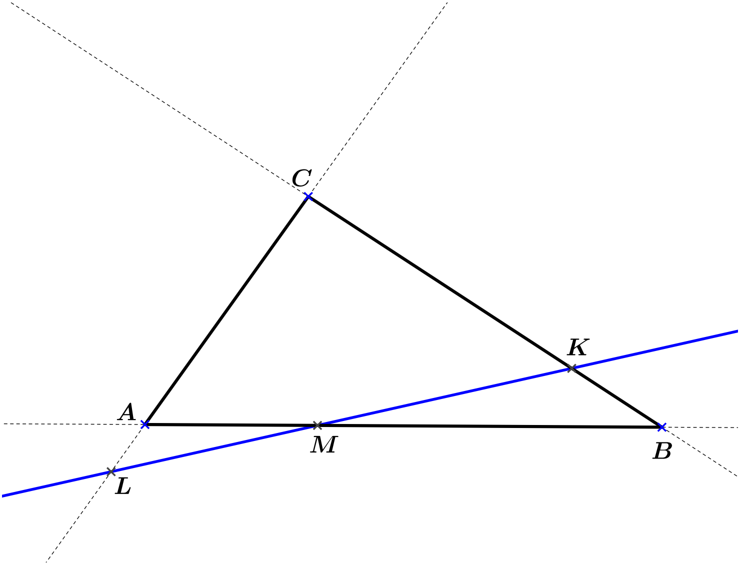
Menelaus's theorem

Theorem (Menelaus's theorem)

If in a triangle ABC with cutting points $K \in \overline{BC}$, $L \in \overline{CA}$, $M \in \overline{AB}$, the points K, L, M are collinear, then we have

$$(A, B; M) \cdot (B, C; K) \cdot (C, A; L) = 1$$

Menelaus’s theorem



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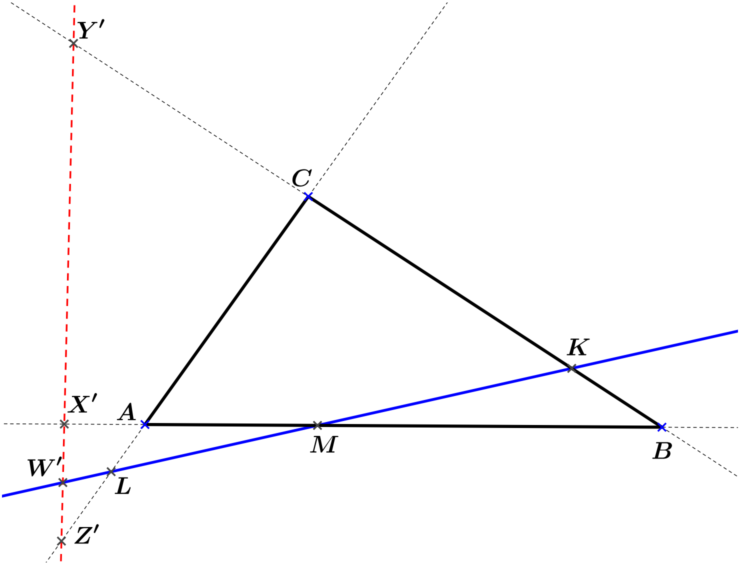

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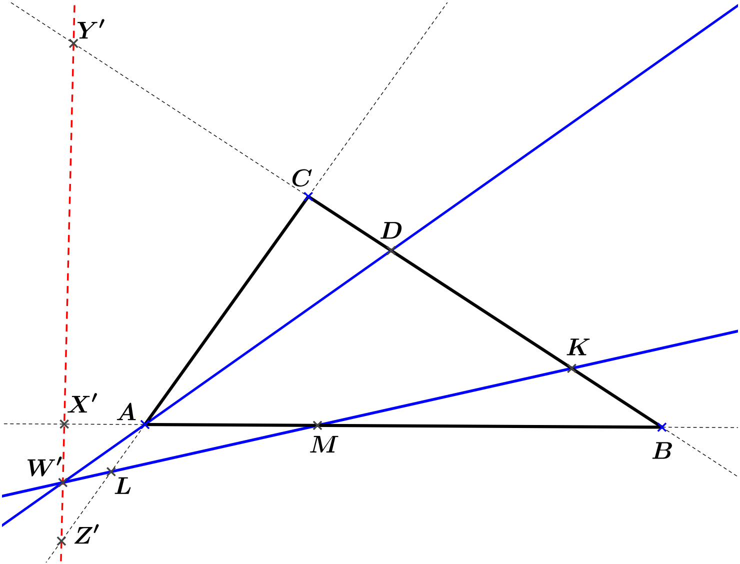

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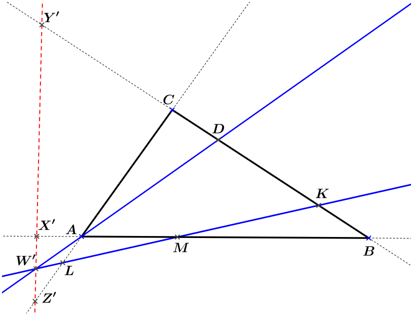
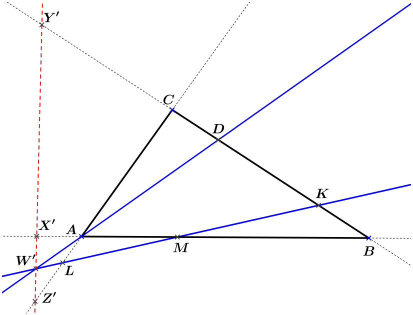


$$(A, B; M) \cdot (B, C; K) \cdot (C, A; L) \stackrel{?}{=} 1$$

Menelaus's theorem

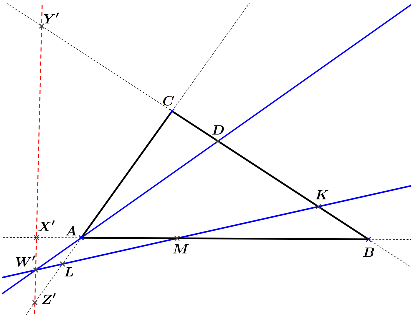


$(A, B; M) \cdot (B, C; K) \cdot (C, A; L) \stackrel{?}{=} 1$

with respect to a line

$(A, B; M, X') \cdot (B, C; K.Y') \cdot (C, A; L, Z') \stackrel{?}{=} 1$

Menelaus's theorem



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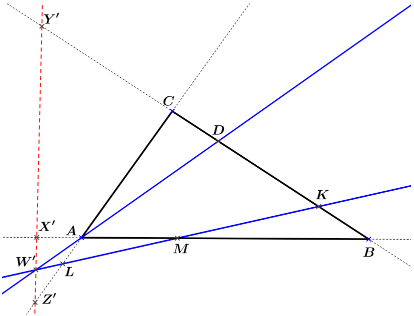
$(A, B; M, X') \cdot (B, C; K.Y') \cdot (C, A; L, Z') \overset{?}{=} 1$

$ABMX' \overset{(W')}{\overline{\overline{\wedge}}} DBKY', CALZ' \overset{(W')}{\overline{\overline{\wedge}}} CDKY'$

Menelaus's theorem



$(A, B; M) \cdot (B, C; K) \cdot (C, A; L) \stackrel{?}{=} 1$

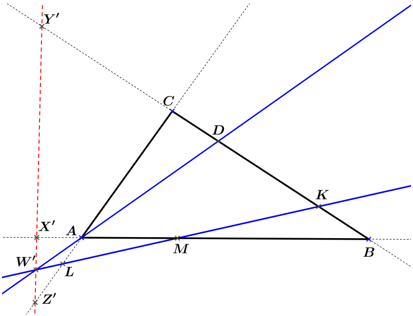
with respect to a line

$(A, B; M, X') \cdot (B, C; K.Y') \cdot (C, A; L, Z') \stackrel{?}{=} 1$

$ABMX' \overset{(W')}{\barwedge} DBKY', CALZ' \overset{(W')}{\barwedge} CDKY'$

$(D, B; K, Y') \cdot (B, C; K, Y') \cdot (C, D; K, Y') \stackrel{?}{=} 1$

Menelaus's theorem



$(A, B; M) \cdot (B, C; K) \cdot (C, A; L) \stackrel{?}{=} 1$

with respect to a line

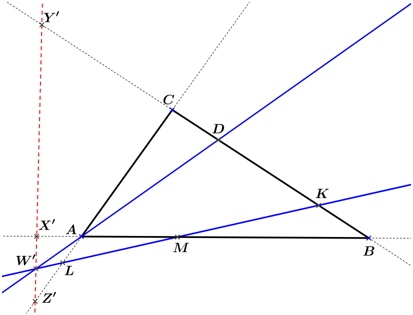
$(A, B; M, X') \cdot (B, C; K.Y') \cdot (C, A; L, Z') \stackrel{?}{=} 1$

$ABMX' \overset{(W')}{\overline{\overline{\wedge}}} DBKY', CALZ' \overset{(W')}{\overline{\overline{\wedge}}} CDKY'$

$(D, B; K, Y') \cdot (B, C; K, Y') \cdot (C, D; K, Y') \stackrel{?}{=} 1$

$$\frac{\overrightarrow{DK} \cdot \overrightarrow{BY}}{\overrightarrow{BK} \cdot \overrightarrow{DY}} \cdot \frac{\overrightarrow{BK} \cdot \overrightarrow{CY}}{\overrightarrow{CK} \cdot \overrightarrow{BY}} \cdot \frac{\overrightarrow{CK} \cdot \overrightarrow{DY}}{\overrightarrow{DK} \cdot \overrightarrow{CY}}$$

Menelaus's theorem



$(A, B; M) \cdot (B, C; K) \cdot (C, A; L) \stackrel{?}{=} 1$

with respect to a line

$(A, B; M, X') \cdot (B, C; K.Y') \cdot (C, A; L, Z') \stackrel{?}{=} 1$

$ABMX' \overset{(W')}{\overline{\overline{\wedge}}} DBKY', CALZ' \overset{(W')}{\overline{\overline{\wedge}}} CDKY'$

$(D, B; K, Y') \cdot (B, C; K, Y') \cdot (C, D; K, Y') \stackrel{?}{=} 1$

$$\frac{\overrightarrow{DK} \cdot \overrightarrow{BY}}{\overrightarrow{BK} \cdot \overrightarrow{DY}} \cdot \frac{\overrightarrow{BK} \cdot \overrightarrow{CY}}{\overrightarrow{CK} \cdot \overrightarrow{BY}} \cdot \frac{\overrightarrow{CK} \cdot \overrightarrow{DY}}{\overrightarrow{DK} \cdot \overrightarrow{CY}} = 1$$

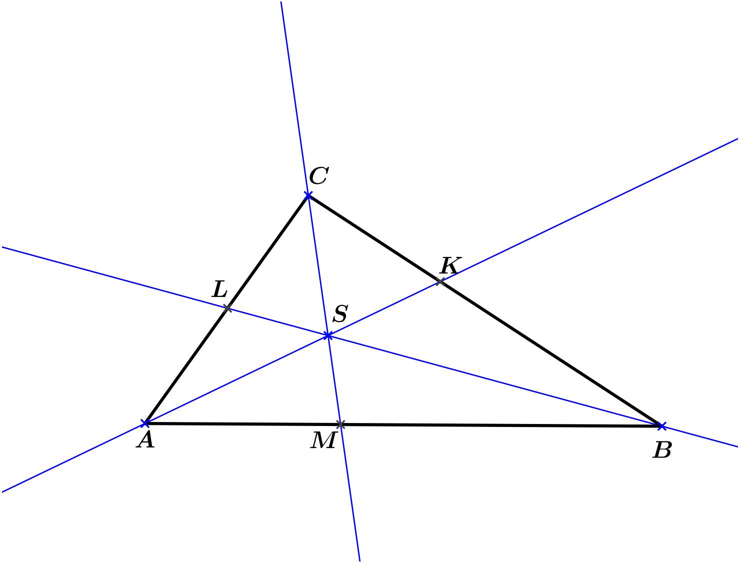
Ceva's theorem

Theorem (Ceva's theorem)

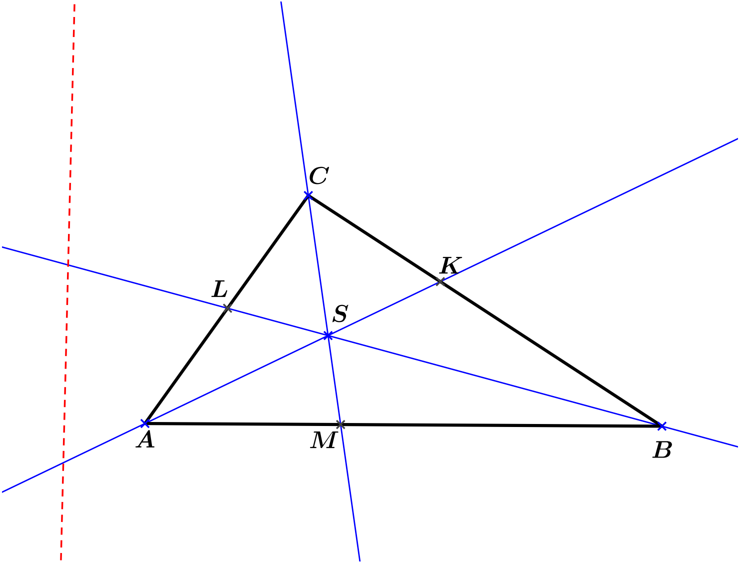
If in a triangle ABC with cutting points $K \in \overline{BC}$, $L \in \overline{CA}$, $M \in \overline{AB}$, *the lines* $\overline{AK}$, $\overline{BL}$, $\overline{CM}$ *are concurrent, then we have*

$$(A, B; M) \cdot (B, C; K) \cdot (C, A; L) = -1$$

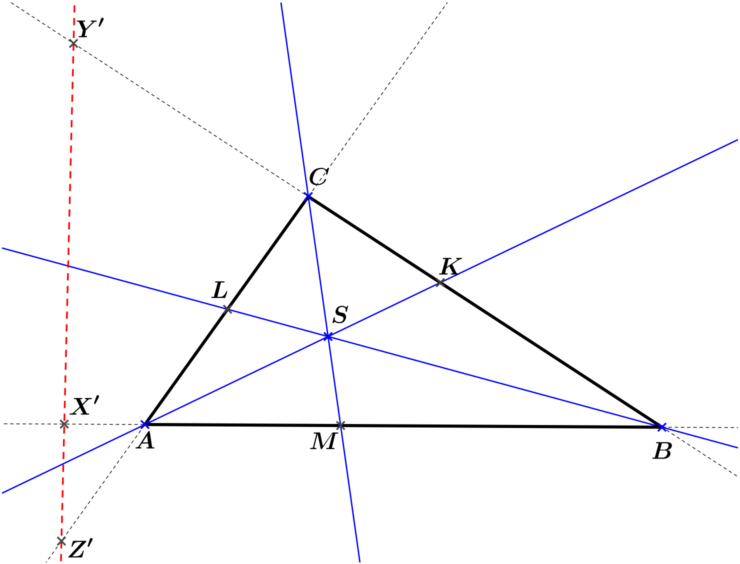
Ceva's theorem



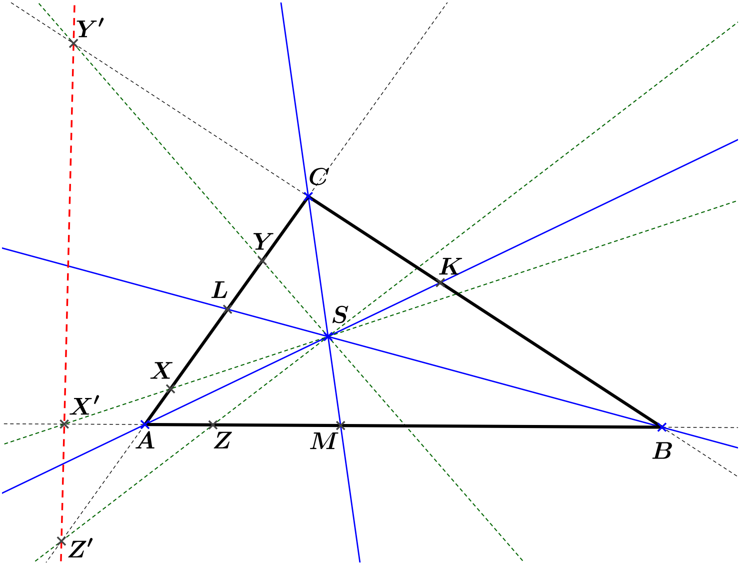
Ceva's theorem



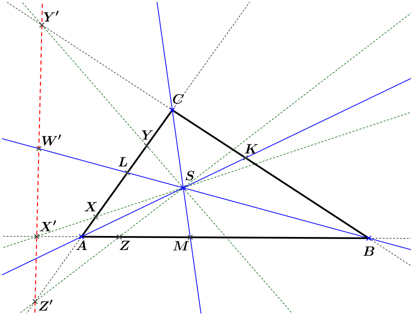
Ceva's theorem



Ceva's theorem

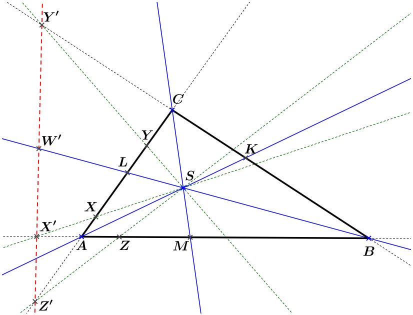


Ceva's theorem



$$(A, B; M) \cdot (B, C; K) \cdot (C, A; L) \stackrel{?}{=} -1$$

Ceva's theorem

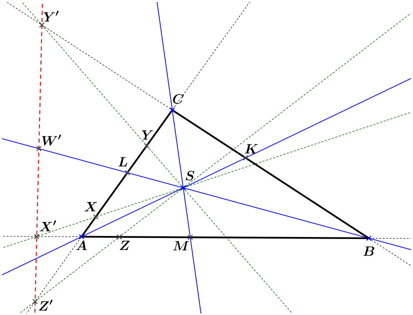


$(A, B; M) \cdot (B, C; K) \cdot (C, A; L) \overset{?}{=} -1$

with respect to a line

$(A, B; M, X') \cdot (B, C; K.Y') \cdot (C, A; L, Z') \overset{?}{=} -1$

Ceva's theorem

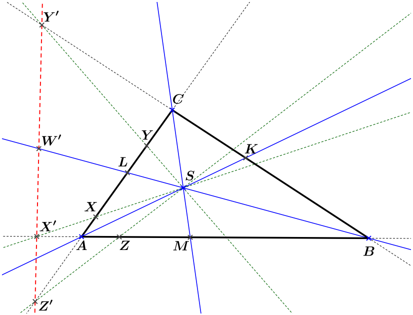


$(A, B; M) \cdot (B, C; K) \cdot (C, A; L) \overset{?}{=} -1$

with respect to a line
$(A, B; M, X') \cdot (B, C; K.Y') \cdot (C, A; L, Z') \overset{?}{=} -1$

$ABMX' \overset{(S)}{\doublebarwedge} ALCX,\ BCKY' \overset{(S)}{\doublebarwedge} LCAY$

Ceva's theorem



$(A, B; M) \cdot (B, C; K) \cdot (C, A; L) \stackrel{?}{=} -1$

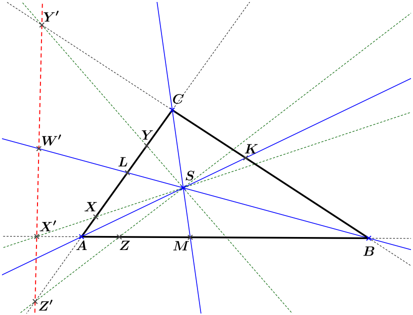
with respect to a line

$(A, B; M, X') \cdot (B, C; K.Y') \cdot (C, A; L, Z') \stackrel{?}{=} -1$

$ABMX' \stackrel{(S)}{\overline{\wedge}} ALCX$, $BCKY' \stackrel{(S)}{\overline{\wedge}} LCAY$

$(A, L; C, X) \cdot (L, C; A.Y) \cdot (C, A; L, Z') \stackrel{?}{=} -1$

Ceva's theorem



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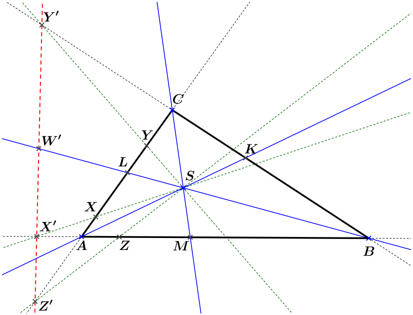
$$ABMX' \stackrel{(S)}{\overline{\overline{\wedge}}} ALCX,\ BCKY' \stackrel{(S)}{\overline{\overline{\wedge}}} LCAY$$

$$(A, L; C, X) \cdot (L, C; A.Y) \cdot (C, A; L, Z') \stackrel{?}{=} -1$$

$$\frac{[SAC]\cdot[SLX]}{[SLC]\cdot[SAX]} \cdot \frac{[SLA]\cdot[SCY]}{[SCA]\cdot[SLY]} \cdot \frac{[SCL]\cdot[SAZ']}{[SAL]\cdot[SCZ']}$$

Ceva's theorem



$(A, B; M) \cdot (B, C; K) \cdot (C, A; L) \stackrel{?}{=} -1$

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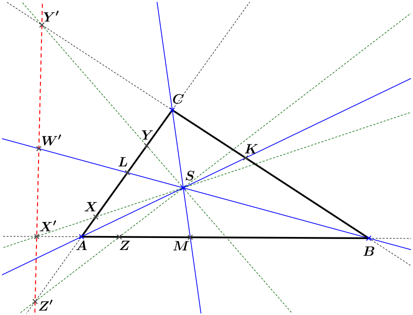
$ABMX' \overset{(S)}{\overline{\overline{\wedge}}} ALCX, BCKY' \overset{(S)}{\overline{\overline{\wedge}}} LCAY$

$(A, L; C, X) \cdot (L, C; A.Y) \cdot (C, A; L, Z') \stackrel{?}{=} -1$

$\frac{[SAC] \cdot [SLX]}{[SLC] \cdot [SAX]} \cdot \frac{[SLA] \cdot [SCY]}{[SCA] \cdot [SLY]} \cdot \frac{[SCL] \cdot [SAZ']}{[SAL] \cdot [SCZ']} \cdot \left(\frac{[SLZ']}{[SLZ']} \right)$



$(A, B; M) \cdot (B, C; K) \cdot (C, A; L) \stackrel{?}{=} -1$

with respect to a line

$(A, B; M, X') \cdot (B, C; K.Y') \cdot (C, A; L, Z') \stackrel{?}{=} -1$

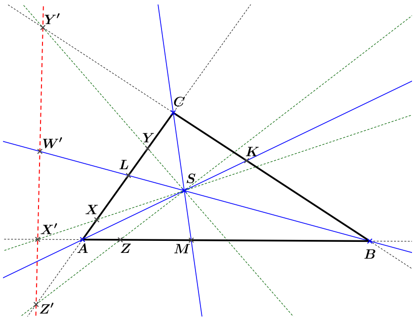
$ABMX' \overset{(S)}{\overline{\wedge}} ALCX, BCKY' \overset{(S)}{\overline{\wedge}} LCAY$

$(A, L; C, X) \cdot (L, C; A.Y) \cdot (C, A; L, Z') \stackrel{?}{=} -1$

$$\frac{[SAC]\cdot[SLX]}{[SLC]\cdot[SAX]} \cdot \frac{[SLA]\cdot[SCY]}{[SCA]\cdot[SLY]} \cdot \frac{[SCL]\cdot[SAZ']}{[SAL]\cdot[SCZ']} \cdot \left(\frac{[SLZ']}{[SLZ']}\right)$$

$$= -\frac{[SLX]\cdot[SAZ']}{[SAX]\cdot[SLZ']} \cdot \frac{[SCY]\cdot[SLZ']}{[SLY]\cdot[SCZ']}$$

Ceva's theorem



$(A, B; M) \cdot (B, C; K) \cdot (C, A; L) \overset{?}{=} -1$

with respect to a line

$(A, B; M, X') \cdot (B, C; K.Y') \cdot (C, A; L, Z') \overset{?}{=} -1$

$ABMX' \overset{(S)}{\overline{\overline{\wedge}}} ALCX,\ BCKY' \overset{(S)}{\overline{\overline{\wedge}}} LCAY$

$(A, L; C, X) \cdot (L, C; A.Y) \cdot (C, A; L, Z') \overset{?}{=} -1$

$$\frac{[SAC]\cdot[SLX]}{[SLC]\cdot[SAX]} \cdot \frac{[SLA]\cdot[SCY]}{[SCA]\cdot[SLY]} \cdot \frac{[SCL]\cdot[SAZ']}{[SAL]\cdot[SCZ']} \cdot \left(\frac{[SLZ']}{[SLZ']}\right)$$

$$= -\frac{[SLX]\cdot[SAZ']}{[SAX]\cdot[SLZ']} \cdot \frac{[SCY]\cdot[SLZ']}{[SLY]\cdot[SCZ']}$$

$$= -\frac{(L,A;X,Z')}{(L,C;Y,Z')}$$

$LAXZ' \overset{(X')}{\overline{\overline{\wedge}}} LBSW';\ LCYZ' \overset{(Y')}{\overline{\overline{\wedge}}} LBSW'$

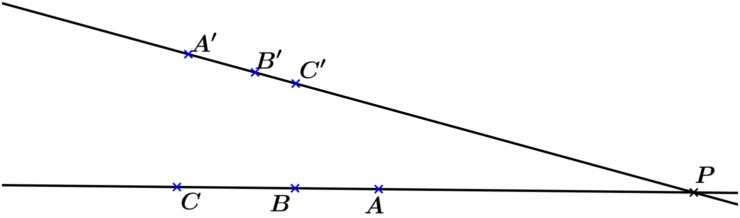
$$= -\frac{(L,B;S,W')}{(L,B;S,W')} = -1$$

Pappus's theorem

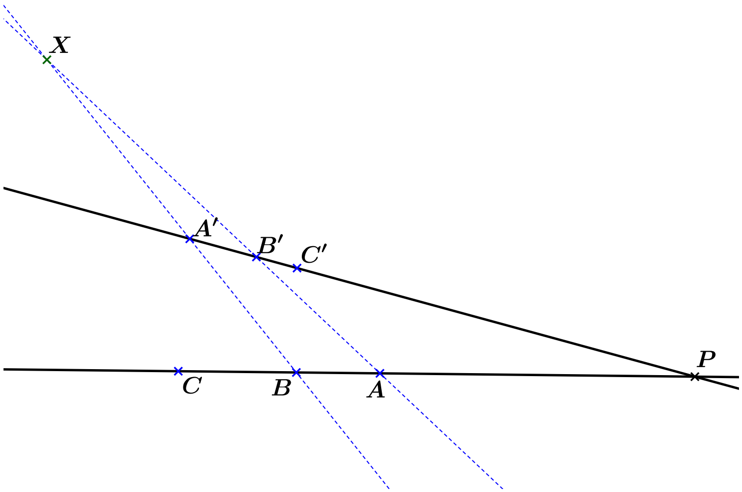
Theorem (Pappus's theorem)

Let A, B, C be three points on a straight line and let A', B', C' be three points on another line. If the lines $\overline{AB'}, \overline{CA'}, \overline{BC'}$ intersect the lines $\overline{BA'}, \overline{AC'}, \overline{CB'}$, respectively, then the three points of intersection are collinear.

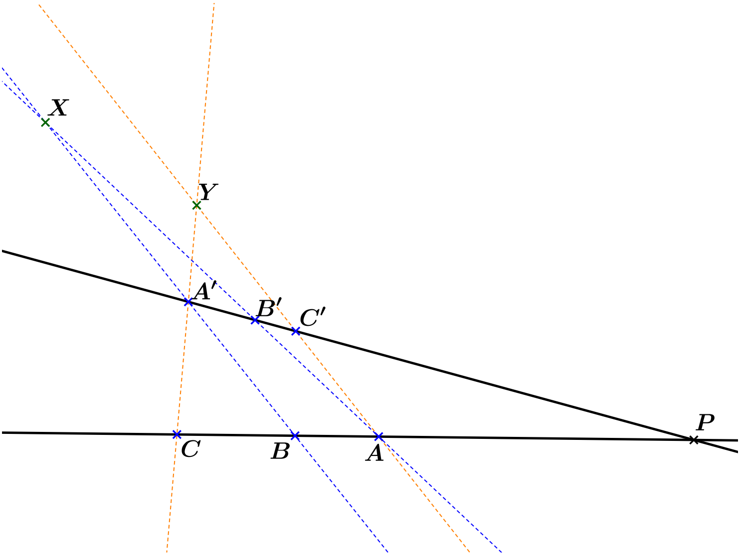
Pappus's theorem



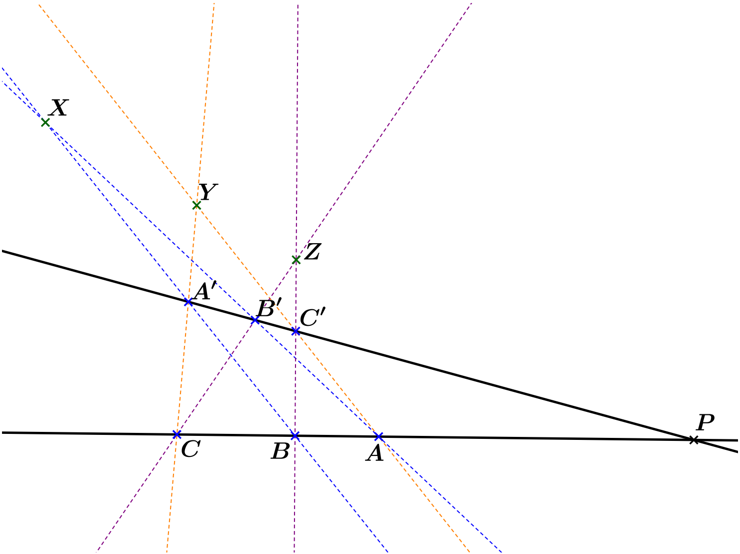
Pappus's theorem



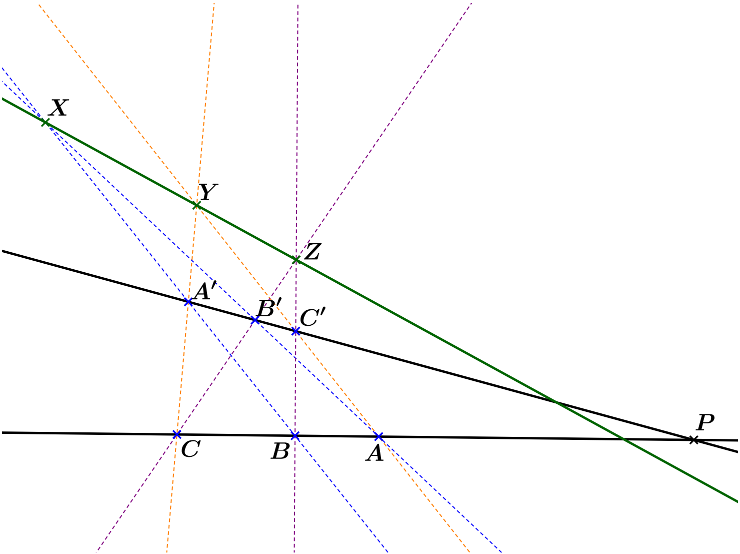
Pappus's theorem



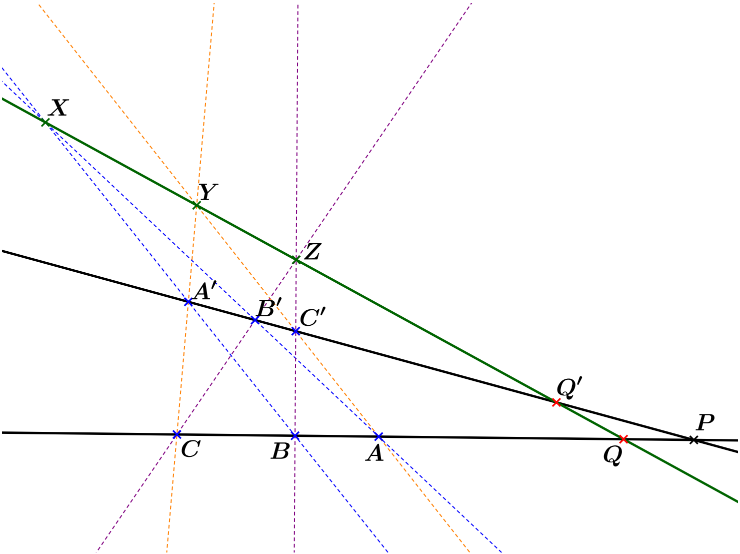
Pappus's theorem



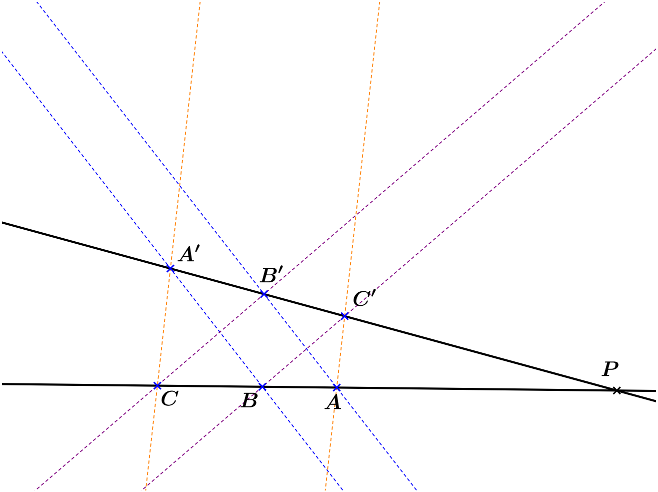
Pappus's theorem



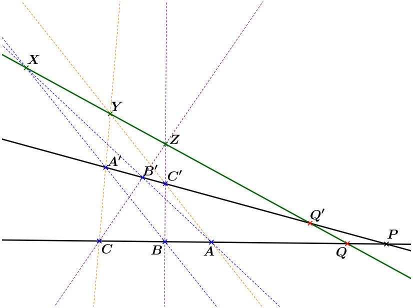
Pappus's theorem



One special easy-to-prove case

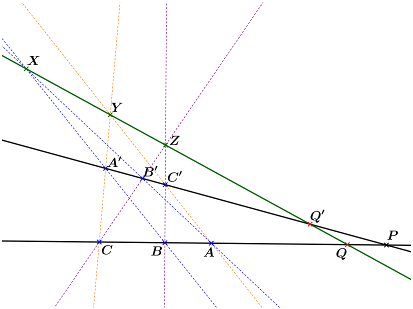


Pappus's theorem



$X = \overline{AB'} \cap \overline{BA'};\ Z = \overline{BC'} \cap \overline{CB'}$

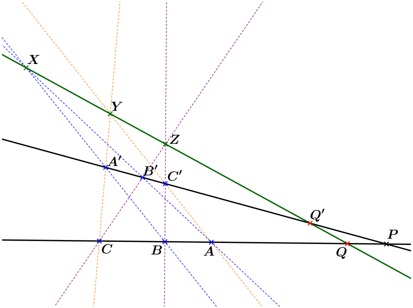
Pappus's theorem



$$X = \overline{AB'} \cap \overline{BA'};\ Z = \overline{BC'} \cap \overline{CB'}$$

$$Y = \overline{AC'} \cap \overline{CA'} \overset{?}{\in} \overline{XZ}$$

Pappus's theorem

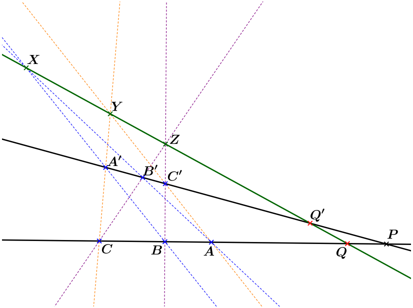


$$X = \overline{AB'} \cap \overline{BA'};\; Z = \overline{BC'} \cap \overline{CB'}$$

$$Y = \overline{AC'} \cap \overline{CA'} \overset{?}{\in} \overline{XZ}$$

$$\text{Let } \overline{AC'} \cap \overline{XZ} = Y_1,\; \overline{CA'} \cap \overline{XZ} = Y_2$$

Pappus's theorem



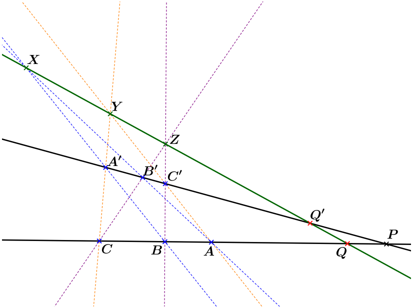
$$X = \overline{AB'} \cap \overline{BA'};\ Z = \overline{BC'} \cap \overline{CB'}$$

$$Y = \overline{AC'} \cap \overline{CA'} \overset{?}{\in} \overline{XZ}$$

$$\text{Let } \overline{AC'} \cap \overline{XZ} = Y_1,\ \overline{CA'} \cap \overline{XZ} = Y_2$$

$$ABPQ \overset{(X)}{\overline{\overline{\wedge}}} B'A'PQ'$$

Pappus's theorem



$$X = \overline{AB'} \cap \overline{BA'};\ Z = \overline{BC'} \cap \overline{CB'}$$

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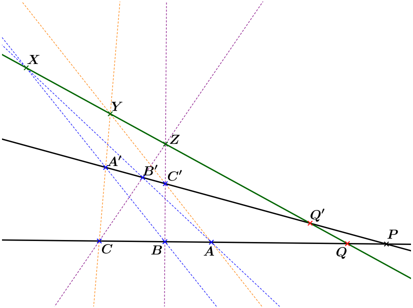
$$ABPQ \overset{(X)}{\overline{\overline{\wedge}}} B'A'PQ'$$

$$ABPQ \overset{(C')}{\overline{\overline{\wedge}}} Y_1ZQ'Q$$

$$B'A'PQ' \overset{(C')}{\overline{\overline{\wedge}}} ZY_2QQ'$$

Pappus's theorem



$$X = \overline{AB'} \cap \overline{BA'};\ Z = \overline{BC'} \cap \overline{CB'}$$

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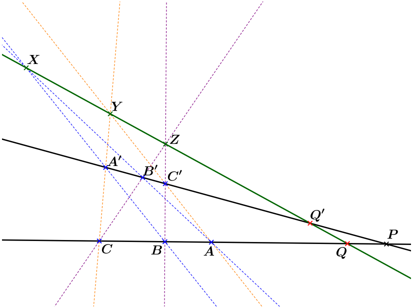
$$ABPQ \overset{(C')}{\overline{\overline{\wedge}}} Y_1ZQ'Q$$

$$B'A'PQ' \overset{(C')}{\overline{\overline{\wedge}}} ZY_2QQ'$$

but $Y_1ZQ'Q \,\overline{\wedge}\, ZY_1QQ'$

Pappus's theorem



$$X = \overline{AB'} \cap \overline{BA'};\ Z = \overline{BC'} \cap \overline{CB'}$$

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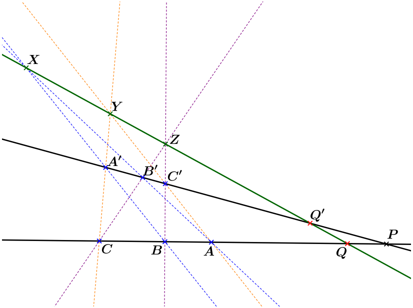
$$B'A'PQ' \stackrel{(C')}{\overline{\overline{\wedge}}} ZY_2QQ'$$

but $Y_1ZQ'Q \barwedge ZY_1QQ'$

therefore $ZY_1QQ' \barwedge ZY_2QQ'$

Pappus's theorem



$$X = \overline{AB'} \cap \overline{BA'};\ Z = \overline{BC'} \cap \overline{CB'}$$

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Let $\overline{AC'} \cap \overline{XZ} = Y_1,\ \overline{CA'} \cap \overline{XZ} = Y_2$

$$ABPQ \overset{(X)}{\overline{\overline{\wedge}}} B'A'PQ'$$

$$ABPQ \overset{(C')}{\overline{\overline{\wedge}}} Y_1ZQ'Q$$

$$B'A'PQ' \overset{(C')}{\overline{\overline{\wedge}}} ZY_2QQ'$$

but $Y_1ZQ'Q \,\overline{\wedge}\, ZY_1QQ'$

therefore $ZY_1QQ' \,\overline{\wedge}\, ZY_2QQ'$

$$Y_1 = Y_2 = Y \in \overline{XZ}$$

References

Paraphrasing Cayley: "*...projective geometry is all geometry.*"

1847 VON STAUDT, K. G. CH.: *Geometrie der Lage*

1856-60 VON STAUDT, K. G. CH.: *Beiträge zur Geometrie der Lage*

1964 COXETER, H. S. M.: *Projective Geometry*

1944-45 HLAVATÝ, V.: *Projektivní geometrie I.-II.*

2011 RICHTER - GEBERT, J.: *Perspectives on Projective Geometry*

Thank you for your attention