ON SETS OF NON-DIFFERENTIABILITY OF LIPSCHITZ AND CONVEX FUNCTIONS

LUDĚK ZAJÍČEK, Praha

Abstract. We observe that each set from the system $\widetilde{\mathcal{A}}$ (or even $\widetilde{\mathcal{C}}$) is Γ-null; consequently the version of Rademacher's theorem (on Gâteaux differentiability of Lipschitz functions on separable Banach spaces) proved by D. Preiss and the author is stronger than that proved by D. Preiss and J. Lindenstrauss. Further we show that the set of non-differentiability points of a convex function on $\mathbb{R}^n$ is σ-strongly lower porous. A discussion concerning sets of Fréchet non-differentiability points of continuous convex functions on a separable Hilbert space is also presented.

Keywords: Lipschitz function, convex function, Gâteaux differentiability, Fréchet differentiability, Γ-null sets, ball small sets, δ-convex surfaces, strong porosity

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1. Introduction

There exist several infinite-dimensional versions of Rademacher's theorem (on Gâteaux differentiability of Lipschitz functions on separable Banach spaces). They assert that a Lipschitz mapping f of a separable Banach space to a Banach space with the Radon-Nikodým property is Gâteaux differentiable "almost everywhere". The version due to Aronszajn [1] (and Phelps [11]) says that the set $NG(f)$ of Gâteaux non-differentiability points of f is null in Aronszajn's (or, equivalently, in Gaussian) sense; see [3]. Further versions of Rademacher's theorem were proved in [16] ($NG(f)$ belongs to the class $\widetilde{\mathcal{A}}$) and in [8] ($NG(f)$ is Γ-null). Since $\widetilde{\mathcal{A}}$ is a proper subsystems of the system $\mathcal{A}$ of Aronszajn's (Gaussian) null sets, the version of [16] is stronger than that of [1]. As shown in [8], the version of [8] is uncomparable with that of [1]. In Section 2 we observe (Theorem 2.4) that the the lemma ([8, Lemma 2.2]) which was used in [8] for proving that $NG(f)$ is Γ-null easily implies that each set from $\widetilde{\mathcal{A}}$ (or from the possibly larger class $\widetilde{\mathcal{C}}$) is Γ-null. Since it is well-known that there exists a Γ-null set which is not in $\mathcal{A}$ (see Remark 2.5), the version of Rademacher's theorem from [16] is stronger than that of [8].

It is also observed (Corollary 2.6) that Theorem 1 together with a deep result of [8] immediately imply that a set belonging to $\mathcal{A}$, constructed in each infinite-dimensional separable superreflexive space X in [9], is not in $\widetilde{\mathcal{A}}$ (or $\widetilde{\mathcal{C}}$). In fact, a set from $\mathcal{A} \setminus \widetilde{\mathcal{A}}$ was constructed in any separable X in [16, Proposition 13], using another more direct methods. However, it is interesting that in superreflexive spaces such a set was already constructed in [9].

If f is a continuous convex function on a separable Banach space X, then the set $NG(f)$ of Gâteaux non-differentiability points of f can be covered by countably many of d.c. (that is delta-convex) hypersurfaces (see [19, Theorem 2] or [3, Theorem 4.20]); moreover, this result is the optimal one, if we are interested in the smallness of sets $NG(f)$ (for a continuous convex f) only. Note that a full characterization of these sets is given in [10] for $X = \mathbb{R}^n$, but even the case of a Hilbert space X is open. In Section 3 we present some consequences of the above mentioned result of [19] and P. Hartman's theorem on superposition of delta-convex mappings. In particular, we show that the set $NG(f)$ is σ-strongly lower porous for each convex function on $\mathbb{R}^n$. The infinite-dimensional case is open; see Problem 1 below. If X^* is separable and f is a continuous convex function on X, then ([2]) the set $NF(f)$ of all Fréchet non-differentiability points is a first category set. This result was strengtened in [14], [15] ($NF(f)$ is σ-porous; even "angle small"). Moreover, in [8] it is proved that $NF(f)$ is Γ-null. As written in [7], p. 32, it is difficult to conjecture a precise result on the nature of the sets $NF(f)$. In Section 3, we briefly discuss this problem and formulate related natural Problem 2.

2. Each set from $\widetilde{\mathcal{C}}$ is Γ-null

First we recall the definitions of the systems $\widetilde{\mathcal{A}}$, $\widetilde{\mathcal{C}}$ from [16].

(The symbol $B(x,r)$ denotes an open ball.)

Definition 2.1. *Let X be a separable Banach space. Then we define the following systems:*

(i) *If $0 \neq v \in X$ and $\varepsilon > 0$, then $\widetilde{\mathcal{A}}(v,\varepsilon)$ is the system of all Borel sets $B \subset X$ such that $\{t : \varphi(t) \in B\}$ is Lebesgue null whenever $\varphi : \mathbb{R} \to X$ is such that the function $x \mapsto \varphi(x) - xv$ has Lipschitz constant at most ε.*

(ii) *$\widetilde{\mathcal{C}}$ is the system of all Borel sets B which are of the form $B = \bigcup_{n=1}^{\infty} B_n$, where $B_n \in \widetilde{\mathcal{A}}(v_n, \varepsilon_n)$ for some $v_n \neq 0$ and $\varepsilon_n > 0$.*

(iii) *$\widetilde{\mathcal{A}}$ is the system of all Borel sets $B \subset X$ such that, if $v_i \neq 0$ are vectors from X with $\overline{\operatorname{span}}(v_i)_{i=1}^{\infty} = X$, then B can be represented as a union $\bigcup_{i=1}^{\infty}\bigcup_{k=1}^{\infty} B(i,k)$, where $B(i,k) \in \widetilde{\mathcal{A}}(v_i, 1/k)$ for all i and k.*

Remark 2.2. (i) Clearly $\widetilde{\mathcal{A}} \subset \widetilde{\mathcal{C}}$ and it is not known whether this inclusion is proper. If $\mathcal{A}$ denotes the system of all Aronszajn's (equivalently, Gaussian) null sets, then trivially $\widetilde{\mathcal{A}} \subset \mathcal{A}$ and it is not difficult to show that $\widetilde{\mathcal{C}} \subset \mathcal{A}$ (see the beginning of the proof of [16, Proposition 13]).

(ii) The property of φ from Definition 2.1 (i) is clearly satisfied (see [16, Remark 5]) if φ is Lipschitz and $\|\varphi'(x) - v\| \leq \varepsilon$ for almost all $x \in \mathbb{R}$. Further, a simple extension argument shows that we obtain the same notion of $\widetilde{\mathcal{A}}(v,\varepsilon)$ if we consider φ defined on an arbitrary closed interval (instead on the whole $\mathbb{R}$); see [16], p. 16.

Now we describe the definition of Γ-null sets from [8], which is based on a sofisticated (but very usefull) combination of the notions of a first category set and of a Lebesgue null set. Let $T := [0,1]^{\mathbb{N}}$ be endowed with the product topology and the product Lebesgue measure μ. Let X be a Banach space and let $\Gamma(X)$ be the space of all continuous mappings $\gamma : T \to X$ having continuous partial derivatives $D_j\gamma$ (with one-sided derivatives at points where the j-th coordinate is 0 or 1). The set $\Gamma(X)$ is equipped by the topology generated by the seminorms $\|\gamma\|_0 = \sup_{t\in T}\|\gamma(t)\|$ and $\|\gamma\|_k = \sup_{t\in T}\|D_k\gamma(t)\|$. (This topology is metrizable by a complete separable metric.) A Borel set $N \subset X$ is called Γ-null if $\mu\{t \in T : \gamma(t) \in N\} = 0$ for residually many $\gamma \in \Gamma(X)$ (i.e., for all γ except a first category set). A possibly non Borel subset of X is called Γ-null if it is contained in a Borel Γ-null set.

Our observation (Theorem 2.4) is an easy consequence of the following fact which is a special case of [8, Lemma 2.2].

Lemma 2.3. *Let $v \in X$ and $\varepsilon > 0$. Then, for residually many $\gamma \in \Gamma(X)$, there exists $k \in \mathbb{N}$ and $c > 0$ such that* $\sup_{t\in T}\|cD_k\gamma(t) - v\| < \varepsilon$.

Theorem 2.4. *If X is separable and $N \in \widetilde{\mathcal{C}}$, then N is Γ-null.*

Proof. Let $v \in X$ and $\varepsilon > 0$ be given. By the definition of $\widetilde{\mathcal{C}}$, it is sufficient to prove that each $B \in \widetilde{\mathcal{A}}(v, \varepsilon)$ is Γ-null. By Lemma 2.3 there exists a residual set $S \subset \Gamma(X)$ such that, for each $\gamma \in S$, there exists $k \in \mathbb{N}$ and $c > 0$ such that

$$\sup_{t \in T} \|cD_k\gamma(t) - v\| < \varepsilon. \tag{1}$$

Choose an arbitrary $\gamma \in S$ and find $k \in \mathbb{N}$ and $c > 0$ such that (1) holds. For each $s = (s_1, s_2, \dots) \in T$, set $p_s(y) := \gamma(s_1, \dots, s_{k-1}, y, s_{k+1}, \dots)$ for $y \in [0, 1]$ and define $\varphi_s(x) := p_s(cx)$ for $x \in [0, 1/c]$. By (1) we have $\|\varphi_s'(x) - v\| < \varepsilon$ for each $x \in [0, 1/c]$. Consequently, using Remark 2.2 (ii), for each $B \in \widetilde{\mathcal{A}}(v, \varepsilon)$ we obtain that $(\varphi_s)^{-1}(B)$ is Lebesgue null; consequently also $(p_s)^{-1}(B) = c \cdot (\varphi_s)^{-1}(B)$ is Lebesgue null. Since γ is continuous and B is Borel, we can use the Fubini theorem for μ and $\gamma^{-1}(B)$ and obtain $\mu(\gamma^{-1}(B)) = 0$, which completes the proof. □

Remark 2.5. As observed in [8], the decomposition result of [9] easily implies that, in the case of an infinite-dimensional superreflexive X, there exists a Γ-null set which is not in $\mathcal{A}$ and thus it is not in $\widetilde{\mathcal{A}}$ (even is not in $\widetilde{\mathcal{C}}$). (Note that clearly $\widetilde{\mathcal{A}} \subset \mathcal{A}$ and also $\widetilde{\mathcal{C}} \subset \mathcal{A}$; see Remark 2.2). Consequently the version of Rademacher's theorem from [16] is stronger than that of [8].

Theorem 2.4 together with a theorem of [8] and the above-mentioned decomposition result of [9] immediately imply the following result.

Corollary 2.6. *In each separable infinite-dimensional superreflexive space X, there exists a set in $\mathcal{A}$ (i.e., a Gaussian null set) which is not in $\widetilde{\mathcal{A}}$ (even is not in $\widetilde{\mathcal{C}}$).*

Proof. By [9], X can be decomposed as a union of a set $A \in \mathcal{A}$ and a set N for which there exists a continuous convex function f on X such that f is Fréchet differentiable at no point of N. We will observe that $A \notin \widetilde{\mathcal{C}}$. Indeed, suppose that $A \in \widetilde{\mathcal{C}}$. Then A is Γ-null by Theorem 2.4. Since N is also Γ-null by [8, Corollary 3.11], we obtain a contradiction. □

(Recall that a set in $\mathcal{A} \setminus \widetilde{\mathcal{A}}$ is constructed in any separable X in [16]; see Introduction.)

3. Every convex function on $\mathbb{R}^n$ is differentiable outside a σ-strongly lower porous set

We start with recalling the notion of σ-porosity (for a resent survey about applications of σ-porosity in Banach spaces see [20]).

Definition 3.1. *Let X be a Banach space and $M \subset X$, $x \in X$, $R > 0$ be given. Then we define $\gamma(x, R, M)$ as the supremum of all $r \geq 0$ for which there exists $z \in X$ such that $B(z, r) \subset B(x, R) \setminus M$. Further define the upper and lower porosity of M at x as*

$$\overline{p}(M, x) := 2 \limsup_{R \to 0+} \frac{\gamma(x, R, M)}{R} \quad \text{and} \quad \underline{p}(M, x) := 2 \liminf_{R \to 0+} \frac{\gamma(x, R, M)}{R}.$$

We say that M is upper (lower, strongly upper, strongly lower) porous at x if $\overline{p}(M, x) > 0$ ($\underline{p}(M, x) > 0$, $\overline{p}(M, x) = 1$, $\underline{p}(M, x) = 1$).

We say that M is upper (lower, strongly upper, strongly lower) porous if M is upper (lower, strongly upper, strongly lower) porous at each point $y \in M$. We say that M is σ-upper (σ-lower, σ-strongly upper, σ-strongly lower) porous if it is a countable union of upper (lower, strongly upper, strongly) porous sets.

Clearly the σ-ideal of all σ-strongly lower porous sets is the smallest from the four just defined σ-ideals.

Another definition we need is the following notion (see [19], [20]) of d.c. (that is delta-convex) surfaces of finite codimension.

Definition 3.2. *Let X be a Banach space and $n \in \mathbb{N}$, $1 \leq n < \dim X$. We say that $A \subset X$ is a d.c.-surface of codimension n if there exist an n-dimensional linear space $F \subset X$, its topological complement E and a d.c. (that is delta-convex) mapping $\varphi : E \to F$ such that $A = \{x + \varphi(x) : \ x \in E\}$. A d.c. surface of codimension 1 will be called a d.c. hypersurface.*

By the smooth part A_s of the surface A we mean the set $A_s = \{x + \varphi(x) : \ x \in S\}$, where S is the set of Gâteaux differentiability points of φ.

For the purpose of this article, we define d.c. surfaces of codimension n (and their smooth parts) also in the singular case $n = \dim X < \infty$ as singletons.

Remark 3.3.
- (i) Note that, since F is finite dimensional, clearly (cf. [18, Corollary 1.8]) φ is a d.c. (delta-convex) mapping if and only if $y^* \circ \varphi$ is a d.c. function (i.e. the difference of two continuous convex functions) for each $y^* \in F^*$ (or equivalently: for each $y^* \in F^*$ from a fixed basis of F^*).
- (ii) It is easy to show (since F is finite-dimensional and φ is locally Lipschitz) that $x \in A_s$ if and only if the (Bouligand) tangent cone of A at x is a closed linear subspace of X of codimension k. Thus A_s does not depend on the choice of E, F and φ.
- (iii) An easy standard argument shows that each (smooth part of a) d.c. surface of codimension k ($k \geq 2$) is a subset of a (smooth part of a) d.c. surface of codimension $k - 1$.
- (iv) If $\dim X < \infty$ then, in the definition of A_s, we can replace Gâteaux differentiability by Fréchet differentiability. Indeed, each d.c. function is locally Lipschitz and a Lipschitz function on a finite-dimensional space is Fréchet differentiable if and only if it is Gâteaux differentiable. Therefore it is easy to see that each d.c. surface A of codimension 1 in $\mathbb{R}^n$ is strongly lower porous at each point of A_s (and thus A_s is a strongly lower porous set).

Lemma 3.4. *Let X be a separable Banach space and $A \subset X$ be a d.c. surface of codimension k ($1 \leq k < \dim X$). Then $A \setminus A_s$ can be covered by countably many d.c. surfaces of codimension $k + 1$.*

Proof. Let F, E, $\varphi : E \to F$ be as in Definition 3.2. By [19, Theorem 2] (or [3, Theorem 4.20]) there exist sets $B_i \subset E$ ($i \in \mathbb{N}$) such that each set B_i is a d.c. surface of codimension 1 in E and φ is Gâteaux differentiable at each point of $E \setminus \bigcup B_i$. Clearly $A \setminus A_s \subset \bigcup_{i \in \mathbb{N}} T_i$, where $T_i := \{e + \varphi(e) : \ e \in B_i\}$.

If $k = \dim X - 1$, then the assertion of the lemma holds, since all B_i and T_i are singletons.

If $k < \dim X - 1$, consider an arbitrary $i \in \mathbb{N}$ and choose a closed linear subspace E_1 of E of codimension 1, $F_1 = \operatorname{span}\{f_1\}$ with $E = E_1 \oplus F_1$ and a d.c. mapping $\varphi_1 : E_1 \to F_1$ such that $B_i = \{u + \varphi_1(u) : \ u \in E_1\}$. It is easy to see that $\varphi_2(u) := u + \varphi_1(u))$, $u \in E_1$, is a locally d.c. mapping from E_1 to E (cf. [18, Lemma 1.7]). Consequently $\varphi_3(u) := \varphi(u + \varphi_1(u))$, $u \in E_1$, is a locally d.c. mapping from E_1 to F by [18, Theorem 4.2]. Therefore $\varphi_4(u) := u + \varphi_1(u) + \varphi(u + \varphi_1(u))$, $u \in E_1$, is a locally d.c. mapping from E_1 to $F_1 \oplus F$. Using separability of E_1, Remark 3.3 (i) and a standard extension procedure for continuous convex functions, it is easy to obtain the assertion of the lemma. □

We will apply the preceding lemma in the case of a finite dimensional X only; in this case we can use Hartman's superposition theorem [5] for d.c. mappings instead of its generalization [18, Theorem 4.2.]. However, the lemma is perhaps of some interest also in the infinite dimensional case.

Proposition 3.5. *Let X be a finite dimensional Banach space and $A \subset X$ be a d.c. surface of codimension k $(1 \leq k < \dim X)$. Then*

(i) *$A \setminus A_s$ can be covered by countably many smooth parts of d.c. surfaces of codimension $k+1$,*
(ii) *A can be covered by countably many smooth parts of d.c. surfaces of codimension k and*
(iii) *A is σ-strongly lower porous.*

Proof. Clearly (i) implies (ii) and (ii) implies (iii) (by Remark 3.3 (iii), (iv)).

If $k = \dim X - 1$ then Lemma 3.4 implies that $A \setminus A_s$ is countable; so (i) trivially holds.

Now suppose that the proposition holds for some $1 < k < \dim X$ and A is a d.c. surface of codimension $k-1$. Then, by Lemma 3.4, $A \setminus A_s$ can be covered by countably many d.c. surfaces of codimension k and thus the induction hypothesis (the condition (ii)) gives that $A \setminus A_s$ can covered by countably many smooth parts of d.c. surfaces of codimension k. Thus we obtain that the proposition holds for $k-1$.

By induction, the proposition follows.

□

Proposition 3.5 and the result of [19] mentioned in Introduction immediately imply the main result of this section (in which, of course, we can write $NF(f)$ instead of $NG(f)$).

Corollary 3.6. *Let X be a finite dimensional Banach space and f be a continuous convex function on X. Then the set $NG(f)$ of all Gâteaux non-differentiability points is σ-strongly lower porous.*

It is not probable that the generalization of Corollary 3.6 holds in an infinite-dimensional space. On the other hand, using [19, Theorem 2] and the fact that every d.c. function is locally Lipschitz, it is easy to see that $NG(f)$ is σ-lower porous whenever f is a continuous convex function on a separable Banach space. However, it seems to be probable that the following problem has a negative answer.

Problem 1 Let f be a continuous convex function on ℓ_2. Is it true that then $NG(f)$ is σ-strongly upper porous?

4. On the set of points where a continuous convex function is not Fréchet differentiable

Suppose that X is an infinite-dimensional Banach space such that X^* is separable (i.e., X is a separable Asplund space) and f is a continuous convex function on X. Denote by $NF(f)$ the set of points at which f is not Fréchet differentiable. Then $NF(f)$ is not only of the first category, but it is also an angle small set ([15], see also [12]) and a Γ-null set [8].

Problem 1 of [15] asks whether $NF(f)$ can be covered by countably many closed convex sets with empty interior and by countably many d.c. hypersurfaces.

This problem was answered in negative by S.V. Konyagin [6] in the case when X is Hilbert. E. Matoušková [9] observed that the negative answer follows from the main decomposition result of [9] in the case when X is superreflexive.

The main aim of the present section is to observe that now the most natural related problem is probably the following.

Problem 2 Let f be a continuous convex function on ℓ_2. Is it true that then $NF(f)$ can be covered by a ball small set and by countably many d.c. hypersurfaces?

Recall the definition of ball small sets:

Definition 4.1. *Let X be a Banach space and $r > 0$. We say that $A \subset X$ is r-ball porous if for each $x \in A$ and $\varepsilon \in (0, r)$ there exists $y \in X$ such that $\|x-y\| = r$ and $B(y, r-\varepsilon) \cap A = \emptyset$. We say that $A \subset X$ is ball small if it can be written in the form $A = \bigcup_{n=1}^{\infty} A_n$, where each A_n is r_n-ball porous for some $r_n > 0$.*

Let us note that:

a) The Hahn-Banach theorem easily implies that, in a Hilbert space, each closed convex set with empty interior is ball small.

b) If X is Hilbert and $M \subset X$ is ball small, then there exists (see [15, Theorem 2] a continuous convex function on X such that $M \subset NF(f)$.

c) Both examples in [6] and [9] give a ball small set which cannot be covered by countably many closed convex sets with empty interior and by countably many d.c. hypersurfaces.

These facts suggest that Problem 2 is very natural. Moreover, we will show that, in this problem, it is not possible to avoid d.c. hypersurfaces. Indeed, in any Hilbert space X with $\dim X \geq 2$, there exists a continuous convex f such that $NF(f)$ is not ball small. This fact, which is mentioned in [15] (without a proof) for $X = \mathbb{R}^n$, follows immediately from Proposition 4.3 below, since for each d.c. hypersurface $C \subset X$ there exists (by [19, Theorem 2]) a continuous convex function on X such that $C \subset NG(f)$.

Before proving Proposition 4.3 we need to recall some (essentially well-known) facts.

Definition 4.2. *Let X be a Banach space, $M \subset X$, $a \in M$ and $r > 0$. We say that M is r-ball supported at a if there exists $z \in X$ such that $\|z - a\| \geq r$ and $B(z, \|z - a\|) \cap M = \emptyset$. We say that M is ball supported at a if it is r-ball supported at a for some $r > 0$.*

Recall that (see e.g. [4]) that $f : (a, b) \to \mathbb{R}$ is called semiconcave (with a linear modulus) if it is locally of the form $f(x) = c(x) + Kx^2$, where $c(x)$ is concave and $K > 0$. A function is called semiconvex, if $-f$ is semiconcave. Each semiconvex function is a d.c. function (since it is locally d.c., see [5]), but a very special one. For example, the second distributional derivative $D^2 f$ of f is clearly a signed Radon measure whose negative part is absolutely continuous with respect to the Lebesgue measure.

We can clearly choose a d.c. function $\varphi : \mathbb{R} \to \mathbb{R}$ such that, for each interval open interval I, neither the positive nor the negative part of $D^2(\varphi|_I)$ is absolutely continuous with respect to the Lebesgue measure. Thus, for each open interval I, the function $\varphi|_I$ is neither semiconvex nor semiconcave. Then we have that

(*) there exists a dense subset D of the graph G of φ such that G is ball supported at no point of D.

Indeed, suppose on the contrary that there exists an open interval I such that G is ball supported at each point of $G \cap (I \times \mathbb{R})$. The Baire Category Theorem easily implies that there exists an open interval $J \subset I$, a dense subset S of $G \cap (J \times \mathbb{R})$ and $r > 0$ such that either

(a) G is r-supported from above at each point $a \in S$ or
(b) G is r-supported from below at each point $a \in S$.

Here "r-supported from above" means that the point z from Definition 4.2 can be chosen to be above G.

If (a) holds, then an easy compactness argument shows that G is r-ball porous from above even at each point of $G \cap (J \times \mathbb{R})$. But this implies (see e.g. [17]) that $\varphi|_J$ is semiconcave, which is a contradiction. Similarly we obtain a contradiction, if (b) holds.

Proposition 4.3. *In every Hilbert space X with $\dim X \geq 2$, there exists a d.c. hypersurface C which is not ball small.*

Proof. Choose a d.c. function $\varphi : \mathbb{R} \to \mathbb{R}$ such that (*) holds.

Further fix unit orthogonal vectors $v, w \in X$, put $Y := \operatorname{span}\{v, w\}^{\perp}$ and then define $C := \{tv + y + \varphi(t)w : \ t \in \mathbb{R},\ y \in Y\}$. It is clear that C is a d.c. hypersurface.

To prove that C is not ball small, suppose on the contrary that $C = \bigcup_{n=1}^{\infty} P_n$, where P_n is r_n-ball porous. Since each d.c. hypersurface is clearly closed, by the Baire Category Theorem we can choose $k \in \mathbb{N}$, $c \in C$ and $\varepsilon > 0$ such that

$$P_k \ \text{ is dense in } \ C \cap B(c, \varepsilon). \tag{2}$$

Let $c = t_0 v + y_0 + \varphi(t_0)w$. Choose $\delta > 0$ so small that $c(t, y) := tv + y + \varphi(t)w \in B(c, \varepsilon/4)$ whenever $t \in I := (t_0 - \delta, t_0 + \delta)$ and $y \in B(y_0, \delta)$.

Now consider an arbitrary open interval $J \subset I$. Using (2), we can choose $t^* \in J$ and $y^* \in B(y_0, \delta)$ such that $c(t^*, y^*) \in P_k \cap B(c, \varepsilon/4)$. Set $r := \min(r_k, \varepsilon/4)$; clearly P_k is r-ball porous. Therefore, for each $j \in \mathbb{N}$ with $1/j < r$, there exists a point $z_j = t_j v + y_j + \tau_j w$ such that $\|z_j - c(t^*, y^*)\| = r$ and $B(z_j, r - 1/j) \cap P_k = \emptyset$. Since clearly $B(z_j, r - 1/j) \subset B(c, \varepsilon)$, (2) implies $B(z_j, r - 1/j) \cap C = \emptyset$.

Since $(z_j - c(t^*, y_j)) \perp Y$, we have $\|z_j - c(t^*, y_j)\| \leq \|(z_j - c(t^*, y_j)) + (y_j - y^*)\| = \|z_j - c(t^*, y^*)\| = r$.

Putting $\widetilde{z}_j := z_j - y_j$, we have $\|\widetilde{z}_j - c(t^*, 0)\| = \|z_j - c(t^*, y_j)\| \leq r$ and $B(\widetilde{z}_j, r - 1/j) \cap C = \emptyset$, since $C = C - y_j$.

Identifying $tv + \tau w$ with (t, w), we have the sequence $\widetilde{z}_j = (t_j, \tau_j) \in \mathbb{R}^2$ which satisfies $\|\widetilde{z}_j - (t^*, \varphi(t^*))\| \leq r$ and $B(\widetilde{z}_j, r - 1/j) \cap G = \emptyset$. Using a standard compactness argument, we obtain that G is r-ball supported at $(t^*, \varphi(t^*))$. Since $J \subset I$ was arbitrary, G is r-ball supported at each point of a dense subset of $G \cap (I \times \mathbb{R})$. Thus, an another simple compactness argument implies that G is r-ball supported at each point of $G \cap (I \times \mathbb{R})$ which contradicts the choice of φ. $\square$

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Authors adress: Luděk Zajíček, Charles University, Faculty of Mathematics and Physics, Sokolovská 83, 186 75 Praha 8, Czech Republic, e-mail: zajicek@karlin.mff.cuni.cz.