CURVES IN BANACH SPACES WHICH ALLOW A C^2 PARAMETRIZATION OR A PARAMETRIZATION WITH FINITE CONVEXITY

JAKUB DUDA AND LUDĚK ZAJÍČEK

Abstract. We give a complete characterization of those $f : [0, 1] \to X$ (where X is a Banach space) which allow an equivalent (smooth) parametrization with finite convexity, and (in the case when X has a Fréchet smooth norm) a C^2 parametrization. For $X = \mathbb{R}$, a characterization for the C^2 case is well-known. However, even in the case $X = \mathbb{R}^2$, several quite new ideas are needed.

1. Introduction

Let X be a (real) Banach space, and a "curve" $f : [a, b] \to X$ be given. There exist several works (e.g. [Z], [C], [T], [BG]; cf. [B] and [GNW]) which investigate conditions under which f allows an equivalent parametrization which is "smooth of the first order" (e.g. it is differentiable, boundedly differentiable, or continuously differentiable). A lot of effort in these articles was also devoted to the (sometimes rather difficult) problem, when we demand that the smooth parametrization has everywhere or almost everywhere non-zero derivative. The above mentioned articles deal with the cases $X = \mathbb{R}^n$ or $X = \mathbb{R}$. These results were generalized (using more or less difficult modifications of the known methods) to the case of an arbitrary X in [DZ] and [D2], where information concerning the history of the "first order case" can be found.

The case of "higher order smooth" parametrizations for $X = \mathbb{R}$ was settled independently by Laczkovich and Preiss [LP] and Lebedev [L]. Both papers contain (formally slightly different) characterizations of those $f : [0, 1] \to \mathbb{R}$ which allow an equivalent C^n ($n \in \mathbb{N}$) parametrization. Lebedev shows that for a continuous $f : [0, 1] \to \mathbb{R}$ there exists a homeomorphism h of $[0, 1]$ onto itself such that $f \circ h$ is a C^n function if and only if

$$\lambda(f(M_f)) = 0 \text{ and } \sum_{\alpha\in A} (\omega^f_\alpha)^{1/n} < \infty, \tag{1.1}$$

where $I_\alpha, (\alpha \in A)$ are all maximal open intervals in $[0, 1]$ on which f is constant or strictly monotone, $M_f := [0, 1] \setminus \bigcup_{\alpha\in A} I_\alpha$ is the set of points "of varying monotonicity" of f, and ω^f_α is the oscillation of f on I_α.

Laczkovich and Preiss show that the same (for a continuous f) holds if and only if

$$V_{1/n}(f, M_f) < \infty, \tag{1.2}$$

2000 *Mathematics Subject Classification.* Primary: 26E20; Secondary: 26A51, 53A04.

Key words and phrases. Curves with finite convexity, C^2 curves, differentiable parametrizations.

where

$$V_{1/n}(f, M_f) := \sup\{\sum_{i=1}^{m} |f(d_i) - f(c_i)|^{1/n}\},$$

the supremum being taken over all systems $[c_i, d_i]$, $i = 1, \dots, m$, of pairwise non-overlapping subintervals of $[0,1]$ with $c_i, d_i \in M_f$. Note that M_f is called K_f in [LP].

Moreover, in [LP] and [L] also the case of $C^{n,\alpha}$ $(0 < \alpha < 1)$ parametrizations is settled.

It appears that the problem of "higher order smooth" parametrizations in the vector case (even for $X = \mathbb{R}^2$) is essentially more difficult than in the case $X = \mathbb{R}$. Roughly speaking, the reason is that in the vector case new problems appear on the "good intervals" $[c, d] \subset [0, 1]$. In the case $X = \mathbb{R}$, the good intervals are those intervals $[c, d]$, on which f is constant or strictly monotone. In the latter case, the arc-length parametrization of f on $[c, d]$ is linear (and thus C^∞ smooth). In the vector case, the good intervals are those intervals $[c, d]$ on which f is either constant or has bounded variation and the arc-length parametrization of f is locally "well-behaved" in (c, d). But now it is a non-trivial problem to find a necessary and sufficient condition, under which f has a parametrization which is "well-behaved" on the interval $[c, d]$. We were able to solve this problem in the cases when "f is well-behaved" means:

(1) f has finite convexity,
(2) f' has bounded variation,
(3) f is C^2 (if X has a Fréchet smooth norm).

Our necessary and sufficient condition in case (3) deals only with $\|F''(x)\|$, where F is an arc-length parametrization of f (i.e., with the first curvature if $X = \mathbb{R}^n$). This result can perhaps be interesting also from the point of view of differential geometry (see Proposition 7.12). The conditions in cases (1) and (2) deal only with the turn (as a function of the length). Note that the turn is a natural generalization of the classical total (integral) curvature to non-smooth curves (see Lemma 2.7). We essentially use properties of curves with finite turn (see Lemma 2.6) which are proved in [P] for $X = \mathbb{R}^3$, in [AR] for $X = \mathbb{R}^n$, and in [D1] for a general X.

Having solved the problem on "good intervals", we were able to solve the "parametrization problem" completely in these three cases. Main Theorems 5.4, 6.4, and 7.7 give characterizations which probably cannot be simplified and can be relatively easily applied (see Remark 5.5(i), Remark 7.8(ii), and Section 8). We also answer the problem of parametrizations with a.e. non-zero derivatives; see Remarks 5.6, 6.5, 7.9. Note that the characterizations in cases (1), (2), and (3) are pairwise different if $\dim X > 1$ (see Examples 6.6, 7.2) but they all coincide for $X = \mathbb{R}$ (see Theorem 4.1). Moreover, (for $\dim X > 1$) in the case when an arc-length parametrization of f is C^2 on the interior of its domain, then the case (1) is equivalent to (2) (which is also true under much weaker assumptions, see Remark 6.7), but the case (2) is not equivalent to (3) (see Example 8.4).

Note that the notion of a function with finite convexity goes back to de la Vallée Poussin (cf. [RV]) and its generalization to the case of vector functions has simple equivalent definitions using variation of a one-sided derivative or the notion of delta-convex mappings (in the sense of [VZ]); see Preliminaries. Namely, the problem of parametrizations with finite convexity (which was motivated by the paper [D1]) was the original problem for us. After solving it, we found that also the (rather close) C^2 problem can be solved using similar methods.

We were not able to solve the vector-valued problem of C^n parametrization for $n \geq 3$. It seems to us that if there is a satisfactory solution of this problem, then it requires some new ideas. On the other hand, our methods give a full characterization of vector-valued curves allowing $C^{1,\alpha}$ $(0 < \alpha \leq 1)$ parametrizations. However, since the proofs are rather technical and lengthy, we plan to publish these results later.

Since there are many technical lemmas, notice that the core of the article is: Lemma 3.4 (for the cases (1) and (2)), Lemma 3.6 (for the case (3)), and the methods of constructions of homeomorphisms in the proofs of Lemma 5.2 and Lemma 7.6.

Finally, note that we do not use any results from [LP] and [L] and that our results easily imply the results on C^2 parametrizations from these articles; see Remark 4.2.

2. Preliminaries

By λ we will denote the Lebesgue measure on $\mathbb{R}$. Throughout the whole article, X will always be a (real) Banach space. By $\mathcal{H}^1$ we will denote the 1-dimensional Hausdorff measure (see e.g. [F]). Let $[a, b] \subset \mathbb{R}$ be an interval. We say that $(x_i)_{i=0}^n$ is a *partition of* $[a, b]$ provided $a = x_0 < x_1 < \cdots < x_n = b$.

A mapping is *L-Lipschitz* provided it is Lipschitz with some constant L (not necessarily the minimal one). If $M \subset A \subset \mathbb{R}$ and $f : A \to X$ are given, then we define the *variation of f on M* as

$$V(f, M) := \sup\{\sum_{i=1}^{n} \|f(x_{i-1}) - f(x_i)\|\},$$

where the supremum is taken over all $(x_i)_{i=0}^n \subset M$ such that $(x_i)_{i=0}^n$ is a partition of $[x_0, x_n]$. We say that $f : [a, b] \to X$ is BV (or *has bounded variation*), provided $V(f, [a, b]) < \infty$. If $f : [a, b] \to X$ is BV, then we define $v_f(x) = V(f, [a, x])$, $x \in [a, b]$. It is well known (see e.g. [K, Theorem 7] together with [F, Theorem 2.10.13]) that if $f : [a, b] \to X$ is Lipschitz, and $f'(x)$ exists for almost all $x \in [a, b]$, then

$$V(f, [a, b]) = \int_a^b \|f'(x)\|\, dx. \tag{2.1}$$

Definition 2.1. (a) Let I be an arbitrary interval, and $f : I \to X$ be continuous. We say that $f^* : J \to X$ is a parametrization of f if there exists an increasing homeomorphism $h : J \to I$ such that $f^* = f \circ h$. We say that f^* is an *arc-length parametrization* if $V(f^*, [u, v]) = v - u$ for every $u, v \in J$.

(b) If $f : [a, b] \to X$ is continuous and BV, then there exists (cf. [F, §2.5.16]) a unique $F : [0, \ell] \to X$ (where $\ell := v_f(b)$) such that $f = F \circ v_f$, and F is 1-Lipschitz. We will reserve the letter F for this *"arc-length parametrization" of f*.

Let I be an arbitrary interval, and $f : I \to X$ be continuous. It is easy to see that f has an arc-length parametrization if and only if $V(f, [a, b]) < \infty$ and $f|_{[a,b]}$ is not constant for each $[a, b] \subset I$. Note that if $f : [a, b] \to X$ is continuous, BV, and not constant on any subinterval of an interval $[\alpha, \beta] \subset [a, b]$, then $G := F|_{[v_f(\alpha), v_f(\beta)]} = f \circ (v_f|_{[\alpha,\beta]})^{-1}$ is an arc-length parametrization of $f|_{[\alpha,\beta]}$.

Let $f : [a, b] \to X$. The derivative f' and the one-sided derivatives $f'_\pm$ are defined in the usual way; at the endpoints we take $f'(a) := f'_+(a)$, and $f'(b) := f'_-(b)$. We say that $f : [a, b] \to X$ *is* C^1 provided $f'(x)$ exists for all $x \in [a, b]$ and f' is continuous on $[a, b]$. We say that $f : [a, b] \to X$ *is* C^2 provided f' is C^1. We say that f *is* $C^{1,BV}$ provided f is C^1, and f' is BV on $[a, b]$. Clearly, if f is C^2, then f is $C^{1,BV}$.

Let $I \subset \mathbb{R}$ be an arbitrary interval and a continuous $f : I \to X$ be given. The *right* (*resp. left*) *unit tangent vector of* f *at* $x \in I$ and the *unit tangent vector of* f *at* $x \in I$ are defined as the limits

$$\tau_+(f, x) = \lim_{t\to 0+} \frac{f(x + t) - f(x)}{\|f(x + t) - f(x)\|}, \quad \tau_-(f, x) = \lim_{t\to 0-} -\frac{f(x + t) - f(x)}{\|f(x + t) - f(x)\|},$$

$$\tau(f, x) = \lim_{\substack{t\to 0\\ x+t\in I}} \operatorname{sgn}(t) \cdot \frac{f(x + t) - f(x)}{\|f(x + t) - f(x)\|},$$

and f is said to be *tangentially smooth* if the function $\tau(f, x)$ is defined and continuous on I. If $f'(x)$ (resp. $f'_+(x)$) exists and is not equal to 0, then clearly $\tau(f, x) = f'(x)/\|f'(x)\|$ (resp. $\tau_+(f, x) = f'_+(x)/\|f'_+(x)\|$).

The notion of “convexity” goes back to de la Vallée Poussin (1908; cf. [RV]).

Definition 2.2. Let X be a normed linear space and $f : [a, b] \to X$ be a continuous mapping. For every partition $D = (x_i)_{i=0}^n$ of $[a, b]$ we put

$$K(f, D) = \sum_{i=1}^{n-1} \left\| \frac{f(x_{i+1}) - f(x_i)}{x_{i+1} - x_i} - \frac{f(x_i) - f(x_{i-1})}{x_i - x_{i-1}} \right\|.$$

The *convexity of* f *on* $[a, b]$ we define as $K(f, [a, b]) = \sup K(f, D)$, where the supremum is taken over all partitions D of $[a, b]$ with $\#D > 2$.

From [VZ, Theorem 2.3] it easily follows that if $f : [a, b] \to X$ is continuous, then $K(f, [a, b]) < \infty$ if and only if $V(f'_+, [a, b)) < \infty$ (see [D1, Lemma 5.5] and Lemma 2.3(ii)). Moreover, it is not difficult to modify the proof of [VZ, Theorem 2.3] to obtain

$$K(f, [a, b]) = V(f'_+, [a, b)) \quad \text{if one of the numbers is finite.} \tag{2.2}$$

For our purposes (since we do not use the classical Definition 2.2), we can consider (2.2) *as the definition of* $K(f, [a, b])$.

Consequently, $C^{1,BV}$ functions are exactly those C^1 functions that have finite convexity. If f is C^2 on $[a, b]$, then (2.1) and (2.2) give

$$K(f, [a, b]) = \int_a^b \|f''(x)\| \, dx. \tag{2.3}$$

Note also (see [D1, Lemma 5.5]) that f has finite convexity if and only if f is a restriction of a mapping $g : (a - \varepsilon, b + \varepsilon) \to X$ which is delta-convex in the sense of [VZ]. We will need the following well-known facts.

Lemma 2.3. *Let $f : [a, b] \to X$ have finite convexity. Then f is Lipschitz and*

(i) *$f'_+(x)$ (resp. $f'_-(x)$) exists at each point of $[a, b)$ (resp. $(a, b]$);*
(ii) *$\lim_{t\to x+} f'_\pm(t) = f'_+(x)$ for $x \in [a, b)$, and $\lim_{t\to x-} f'_\pm(t) = f'_-(x)$ for $x \in (a, b]$;*
(iii) *the set $A := \{x \in (a, b) : f'_+(x) \neq f'_-(x)\}$ is countable and $\sum_{x\in A} \|f'_+(x) - f'_-(x)\| < \infty$;*

(iv) *if $X = \mathbb{R}$, then for each $x \in M_f \cap (a,b)$ we have $f'_-(x) \cdot f'_+(x) \le 0$ (where M_f is the set of points of varying monotonicity of f; see Introduction);*
(v) *if $[c,d] \subset [a,b)$, then $V(f'_+, [c,d]) = K(f, [c,d]) + \|f'_+(d) - f'_-(d)\|$.*

Proof. We see that f is Lipschitz by e.g. [V, Proposition 2.3]. For the part (i), see [V, Proposition 2.4]. Part (ii) follows from [VZ, Note 3.2] and the fact mentioned just before the lemma. (We can also use (2.2) to prove that f'_+ has one-sided limits and then use the mean value theorem [B, Chap. I, par. 2, Proposition 3].) Concerning part (iii), note that if $x \in A$, then $f'_-(x) = \lim_{t\to x-} f'_+(t)$ by part (ii), and thus we easily obtain

$$\sum_{x\in A} \|f'_+(x) - f'_-(x)\| \le V(f'_+, [a,b)) = K(f, [a,b]) < \infty.$$

To prove the part (iv), for a contradiction, suppose $\min(f'_+(x), f'_-(x)) > 0$ (the other case $\max(f'_+(x), f'_-(x)) < 0$ can be handled by taking $-f$ instead of f). By (ii) there exists $\delta > 0$ such that $f'_+(y) > 0$ for all $y \in (x-\delta, x+\delta)$. Since f is Lipschitz, we have $f(t) - f(s) = \int_s^t f'_+(\xi)\, d\xi > 0$ for all $x - \delta < s < t < x + \delta$, which contradicts $x \in M_f$. Part (v) easily follows from (2.2) and part (ii). $\square$

Let $f : [a,b] \to X$ be continuous, and suppose that $\tau_+(f,x)$ exists for all $x \in [a,b)$. We define the (*tangential*) *turn* $T_a^b f$ *of* f *on* $[a,b]$ as

$$T(f,[a,b]) = T_a^b f := V(\tau_+(f,x), [a,b)).$$

We say that f *has finite turn on* $[a,b]$ provided $T_a^b f < \infty$. If $f : G \to X$, where $G \subset \mathbb{R}$ is open, then we say that f *has locally finite turn on* G provided $T(f,[a,b]) < \infty$ for each interval $[a,b] \subset G$.

Remark 2.4. In [P], [AR], and [D1], several formally different definitions of turn are considered. The results from [D1] imply that in case X is a Hilbert space, all these quantities are "almost equal" and so we could use any of them. However, only in [D1] is the turn investigated in the context of a general Banach space. In [D1], the (tangential) turn (for a general Banach space) is defined as $V(g,[a,b])$, where $g(x) := \tau_+(f,x)$ for $x \in [a,b)$, and $g(b) := \tau_-(f,b)$. Lemma 2.5 implies that this definition gives the same notion of finite turn as our current definition.

Lemma 2.5. *Let $f : [a,b] \to X$ be continuous and $T^* := V(\tau_+(f,\cdot), (a,b)) < \infty$. Then $\tau_+(f,a)$ and $\tau_-(f,b)$ exist and*

$$T(f,[a,b]) = T^* = V(g,[a,b]), \tag{2.4}$$

where $g(x) := \tau_+(f,x)$ for $x \in [a,b)$ and $g(b) := \tau_-(f,b)$.

Proof. Since $T^* < \infty$, it is easy to see that $\lim_{\varepsilon\to 0+} \operatorname{diam}(\{\tau_+(f,x) : x \in (a,a+\varepsilon)\}) = 0$. Using completeness of X, we obtain that $\lim_{x\to a+} \tau_+(f,x) =: w$ exists. By [D1, Theorem 3.5], we obtain $\tau_+(f,a) = w$. Similarly, we obtain that $\tau_-(f,b) = \lim_{x\to b-} \tau_+(f,x)$ exists. Now the proof of (2.4) is easy. $\square$

Suppose that $f : [a,b] \to X$ is continuous and h is an increasing homeomorphism of $[c,d]$ onto $[a,b]$. Since clearly $\tau_+(f\circ h, t) = \tau_+(f, h(t))$, we see that

$$T(f\circ h, [c,d]) = T(f,[a,b]), \tag{2.5}$$

if one side of this equality is defined. We will need the following properties of curves with finite turn.

Lemma 2.6. *Let $f : [a, b] \to X$ be a continuous function with finite turn on $[a, b]$. Then f is BV and is not constant on any interval. Let $f_1 : [c, d] \to X$ be an arc-length parametrization of f, and let $\eta : [c, d] \to [a, b]$ be the increasing homeomorphism for which $f_1 = f \circ \eta$. Then:*

(i) *$(f_1)'_+(x)$ (resp. $(f_1)'_-(x)$) exists for all $x \in [c, d)$ (resp. $x \in (c, d]$), and $(f_1)'_+(x) = \tau_+(f, \eta(x))$ (resp. $(f_1)'_-(x) = \tau_-(f, \eta(x))$) at all such points.*
(ii) *$K(f_1, [c, d]) = T(f, [a, b]) < \infty$.*
(iii) *$\tau_+(f, \cdot)$ is continuous from the right at all $x \in [a, b)$.*
(iv) *If $a < \xi < b$, then $T(f, [a, b]) = T(f, [a, \xi]) + T(f, [\xi, b]) + \|\tau_+(f, \xi) - \tau_-(f, \xi)\|$.*
(v) *If $\tau(f, x)$ exists for all $x \in [a, b]$, then f is tangentially smooth on $[a, b]$.*

Proof. The function f is BV by e.g. [D1, Corollary 3.4 and Lemma 4.4(ii)]. Clearly, the function f is not constant on any subinterval of $[a, b]$. Part (i) follows from [D1, Lemma 4.5(i)]. Part (ii) follows from (i) and (2.2). Part (iii) follows from (i) and Lemma 2.3(ii). For part (iv), by (i),(ii), (2.5), and additivity of variation, we have

$$T(f, [a, b]) = V((f_1)'_+, [c, d)) = V((f_1)'_+, [c, u]) + V((f_1)'_+, [u, d)),$$

where $u = \eta^{-1}(\xi)$. By (ii), f_1 has finite convexity, and thus by Lemma 2.3(v) we obtain

$$V((f_1)'_+, [c, u]) = K(f_1, [c, u]) + \|(f_1)'_+(u) - (f_1)'_-(u)\|.$$

By (ii), we have that $K(f_1, [c, u]) = T(f, [a, \xi])$ and the conclusion follows since $V((f_1)'_+, [u, d)) = T(f, [\xi, b])$, and $\tau_\pm(f, \xi) = \tau_\pm(f_1, u) = (f_1)'_\pm(u)$ by (i). Part (v) follows by (i), (ii) and Lemma 2.3(ii). $\square$

Lemma 2.7. *Let $f : [a, b] \to X$ be continuous and $f_1 : [c, d] \to X$ be an arc-length parametrization of f. If f_1 is C^2 smooth, then*

$$T(f, [a, b]) = \int_c^d \|f_1''(x)\|\, dx.$$

Proof. By (2.1) and the fact that f_1 is 1-Lipschitz it follows that $\|f_1'\| \le 1$ and

$$d - c = V(f_1, [c, d]) = \int_c^d \|f_1'(x)\|\, dx.$$

Thus $\|f_1'\| = 1$ almost everywhere in $[c, d]$. Since f_1 is C^2, it follows that $\|f_1'(x)\| = 1$ for all $x \in [c, d]$, and thus $f_1'(x) = \tau(f_1, x)$ for all $x \in [c, d]$. Now, the conclusion follows from (2.1) applied to f_1' (note that f_1' is Lipschitz since $(f_1')' = (f_1)''$ is bounded). $\square$

Lemma 2.8. *Suppose that $f : [a, b] \to X$ is a continuous function with locally finite turn in (a, b). Let $\mu \in (a, b)$, $g(x) := T(f, [\mu, x])$ for $x \in (\mu, b)$ and $h(x) := T(f, [x, \mu])$ for $x \in (a, \mu)$. Then*

(i) *g is continuous from the left at each $x \in (\mu, b)$ and $g(\mu+) = 0$,*
(ii) *$|g(x+) - g(x)| \le 2$ for each $x \in (\mu, b)$,*
(iii) *h is continuous from the right at each $x \in (a, \mu)$, $h(\mu-) = 0$, and*
(iv) *$|h(x-) - h(x)| \le 2$ for each $x \in (a, \mu)$.*

Proof. Since $g(t) = V(\tau_+(f, \cdot), (\mu, t))$ for $\mu < t < b$ and $h(t) = V(\tau_+(f, \cdot), (t, \mu))$ for $a < t < \mu$ (by Lemma 2.5), (i) and (iii) follow easily from the definition of variation. Let $\mu < x < t < b$. Then by Lemma 2.6(iv), $g(t) = g(x) + \|\tau_+(f, x) - \tau_-(f, x)\| +$

$V(\tau_+(f,\cdot),(x,t))$. By definition of variation, $\lim_{t\to x+} V(\tau_+(f,\cdot),(x,t)) = 0$, so we have $|g(x+) - g(x)| = \|\tau_+(f,x) - \tau_-(f,x)\| \le 2$ and part (ii) follows. As for $a < t < x < \mu$, we have $h(t) = V(\tau_+(f,\cdot),(t,x)) + \|\tau_+(f,x) - \tau_-(f,x)\| + h(x)$, the proof of (iv) is analogous. □

Let now $f : [0,1] \to X$ with finite convexity and $[a,b] \subset [0,1]$ be given. Then, only for the use in the present paper, we will define

$$K^*(f,[a,b]) := \|f'_-(a) - f'_+(a)\| + K(f,[a,b]) + \|f'_-(b) - f'_+(b)\|, \tag{2.6}$$

where we put $f'_-(0) := f'_+(0)$ and $f'_+(1) := f'_-(1)$. Using (2.2) and Lemma 2.3(iii), it is easy to verify that

$$K^*(f) := \sup\{\sum_{I\in\mathcal{S}} K^*(f,I)\} < \infty, \tag{2.7}$$

where the supremum is taken over all finite systems $\mathcal{S}$ of closed pairwise non-overlapping subintervals of $[0,1]$.

3. Lemmas

Lemma 3.1. *Let $f : [0,1] \to X$ have finite convexity. Let $C := \{x \in [0,1] : f'_+(x) = 0$ or $f'_-(x) = 0\}$. Then $\mathcal{H}^1(f(C)) = 0$.*

Proof. Denote $C^* := \{x \in [0,1] :\ f'(x) = 0\}$. Since f is Lipschitz, [K, Theorem 7] implies $\mathcal{H}^1(f(C^*)) = 0$. Since $C\setminus C^*$ is countable by Lemma 2.3(i),(iii), the assertion follows. □

Lemma 3.2. *Let $f : [a,b] \to X$ be continuous. Let $\emptyset \ne G \subset (a,b)$ be an open set, $H := [a,b] \setminus G$ and (a_t,b_t), $t \in T$, be all (pairwise different) components of G. Then:*

- (i) *If $\mathcal{H}^1(f(H)) = 0$, then $V(f,[a,b]) = \sum_{t\in T} V(f,[a_t,b_t])$.*
- (ii) *If $V(f,[a,b]) = \sum_{t\in T} V(f,[a_t,b_t]) < \infty$, then $\mathcal{H}^1(f(H)) = 0$.*
- (iii) *If $\mathcal{H}^1(f(H)) = 0$ and f is L-Lipschitz on each $[a_t,b_t]$, then f is L-Lipschitz on $[a,b]$.*
- (iv) *If f is BV and $\mathcal{H}^1(f(H)) = 0$, then $\lambda(v_f(H)) = 0$.*

Proof. Part (i) is an easy consequence of the vector form ([F, Theorem 2.10.13.]) of Banach's indicatrix theorem; see [DZ, Lemma 2.7]. For the part (ii), let $\varepsilon > 0$ and choose a finite $S \subset T$ such that $\sum_{t\in S} V(f,[a_t,b_t]) > V(f,[a,b]) - \varepsilon$. Then by [F, Corollary 2.10.12] we see that

$$\mathcal{H}^1(f(H)) \le \sum_{i=1}^{N} \mathcal{H}^1(f([c_i,d_i])) \le \sum_{i=1}^{N} V(f,[c_i,d_i]) \le \varepsilon,$$

where $[c_i,d_i]$, $i = 1,\dots,N$, are components of $[a,b] \setminus \bigcup_{t\in S}(a_t,b_t)$. Therefore $\mathcal{H}^1(f(H)) = 0$.

For part (iii), consider arbitrary $c,d \in H$, $c < d$. Applying (i) to f on $[c,d]$, we obtain

$$\begin{aligned}|f(d) - f(c)| \le V(f,[c,d]) &= \sum\{V(f,[a_t,b_t]) : t \in T, [a_t,b_t] \subset [c,d]\}\\ &\le \sum\{L|b_t - a_t| : t \in T, [a_t,b_t] \subset [c,d]\} \le L(d-c),\end{aligned}$$

where we used the fact that the variation of an L-Lipschitz function on an interval is also L-Lipschitz. Since f is L-Lipschitz on H and on each $[a_t, b_t]$, it is clearly L-Lipschitz on $[a,b]$. Part (iv) easily follows from (i). $\square$

The following easy inequality is well known (see e.g. [MS, Lemma 5.1]):

$$\text{if } u, v \in X \setminus \{0\}, \text{ then } \left\| \frac{u}{\|u\|} - \frac{v}{\|v\|} \right\| \leq \frac{2}{\|u\|} \|u - v\|. \tag{3.1}$$

Lemma 3.3. *Let $I = [a,b] \subset [0,1]$ and let $f : I \to X$ have finite convexity on I. Define $i := \inf_{x \in [a,b)} \|f'_+(x)\|$, and $S := \sup_{x \in [a,b)} \|f'_+(x)\|$. Then*

(i) *$S - i \leq K(f, I)$,*
(ii) *$V(f, I) \leq S \cdot \lambda(I)$, and*
(iii) *if $i > 0$, then $i \cdot T(f, I) \leq 2 \cdot K(f, I)$.*

Proof. To prove (i), observe that for each $\varepsilon > 0$ we can choose $p, q \in I$ such that $S - i < \|f'_+(q) - f'_+(p)\| + \varepsilon$. Thus the conclusion follows by the equality (2.2).

Since f is Lipschitz (see Lemma 2.3), by (2.1) and Lemma 2.3(iii) we obtain $V(f, I) = \int_a^b \|f'_+(t)\| \, dt \leq S \cdot \lambda(I)$.

Finally, for part (iii), consider arbitrary points $x_0 < \cdots < x_m$ in $[a,b)$. Applying (3.1) to $u = f'_+(x_j)$ and $v = f'_+(x_{j+1})$, we obtain $\|\tau_+(f, x_{j+1}) - \tau_+(f, x_j)\| \leq \frac{2}{i} \|f'_+(x_{j+1}) - f'_+(x_j)\|$, so (2.2) implies (iii). $\square$

Our first basic lemma is the following ($K^*(f, I)$ is defined in (2.6)):

Lemma 3.4. *Let $f : [0,1] \to X$ have finite convexity and $I = [a,b] \subset [0,1]$.*

(i) *If $\inf\{\|f'_+(x)\| : x \in I \setminus \{1\}\} = 0$ or $\inf\{\|f'_-(x)\| : x \in I \setminus \{0\}\} = 0$, then we have $\sqrt{V(f, I)} \leq \frac{1}{2} K^*(f, I) + \frac{1}{2} \lambda(I)$.*
(ii) *If $0 < \delta \leq T(f, I) < \infty$, then $\sqrt{V(f, I)} \leq \frac{2+\delta}{2\delta} K^*(f, I) + \frac{1}{2} \lambda(I)$.*

Proof. Denote $K^* := K^*(f, I)$ and put $f'_-(0) := f'_+(0)$ and $f'_+(1) := f'_-(1)$. For the part (i), let $\varepsilon > 0$ and $w \in I$ be such that $\|f'_+(w)\| < \varepsilon$ or $\|f'_-(w)\| < \varepsilon$. Distinguishing the cases $w = a$, $w = b$ and $w \in (a,b)$, by Lemma 2.3(ii) we can clearly find a point $w^* \in [a,b)$ with $\|f'_+(w^*)\| < \varepsilon + \|f'_+(a) - f'_-(a)\| + \|f'_+(b) - f'_-(b)\|$. Thus, also using (2.2), we obtain $\|f'_+(x)\| \leq \|f'_+(x) - f'_+(w^*)\| + \|f'_+(w^*)\| \leq K^* + \varepsilon$ for each $x \in [a,b)$. Thus by (2.1) we have

$$\sqrt{V(f, I)} = \sqrt{\int_I \|f'_+(x)\| \, dx} \leq \sqrt{\lambda(I)(K^* + \varepsilon)} \leq \frac{1}{2}(K^*(f, I) + \varepsilon + \lambda(I)).$$

Now finish the proof by sending $\varepsilon \to 0$.

To prove the part (ii), we can assume that $\inf\{\|f'_+(x)\| : x \in [a,b)\} =: i > 0$ (otherwise the conclusion follows by part (i)). Let $S := \sup\{\|f'_+(x)\| : x \in [a,b)\}$. By Lemma 3.3(iii), it follows that $i \cdot \delta \leq i \cdot T(f, I) \leq 2 \cdot K(f, I)$, and by part (i) of the same lemma that $S - i \leq K(f, I)$. It follows that $S \leq S - i + i \leq \left(1 + \frac{2}{\delta}\right) K(f, I)$, and because by Lemma 3.3(ii), we have $V(f, I) \leq S \cdot \lambda(I)$. Finally, we obtain

$$\sqrt{V(f, I)} \leq \frac{1}{2}\left(S + \frac{V(f, I)}{S}\right) \leq \frac{1}{2}\left(\left(1 + \frac{2}{\delta}\right) K(f, I) + \lambda(I)\right).$$

$\square$

We prove the following lemma without the assumption of smoothness of the norm of X (under this assumption the proof would be more straightforward).

Lemma 3.5. *Let I be a closed interval and $f : I \to X$ be such that $\|f'(x)\| \geq i > 0$ and $\|f''(x)\| \leq M$ for each $x \in I$. Let $F = f \circ v_f^{-1}$. Then $\|F''(y)\| \leq \frac{2\,M}{i^2}$ for each $y \in v_f(I)$ at which $F''(y)$ exists.*

Proof. Let $x, y \in I$. Since f' is M-Lipschitz, by (3.1) we have

$$\left\| \frac{f'(x)}{\|f'(x)\|} - \frac{f'(y)}{\|f'(y)\|} \right\| \leq \frac{2}{\|f'(x)\|} \|f'(x) - f'(y)\| \leq \frac{2\,M}{i} |x - y|.$$

Consequently, $g := \frac{f'}{\|f'\|}$ is $\frac{2M}{i}$-Lipschitz. Since (2.1) implies $v_f' = \|f'\|$, we have $(v_f^{-1})' = \frac{1}{\|f' \circ v_f^{-1}\|}$. Thus, v_f^{-1} is $\frac{1}{i}$-Lipschitz; so $F' = (f \circ v_f^{-1})' = g \circ v_f^{-1}$ is $\frac{2M}{i^2}$-Lipschitz on $v_f(I)$, and the conclusion follows. $\square$

Our second basic lemma is the following:

Lemma 3.6. *Let I be a closed interval, $f : I \to X$ be C^2 on I. Suppose that f is not constant on any subinterval of I and $F := f \circ v_f^{-1}$ is C^2 on $v_f(\mathrm{int}(I))$. Set $L := \sup_{y \in v_f(\mathrm{int}(I))} \|F''(y)\|$, $M := \sup_{x \in I} \|f''(x)\|$, $i := \inf_{x \in I} \|f'(x)\|$, and suppose that $i > 0$ and $L \cdot V(f, I) \geq \delta > 0$. Then*

$$\sqrt{V(f, I)} \leq \sqrt{M} \left(\frac{\sqrt{\delta}}{2} + \frac{2}{\sqrt{\delta}} \right) \lambda(I).$$

Proof. Observe that $i^2 L \leq 2M$ by Lemma 3.5; so $M > 0$. Let $S := \sup_{x \in I} \|f'(x)\|$, and $q := \frac{1}{2}\sqrt{\frac{\delta}{M}}$. By Lemma 3.3(ii) we have

$$\sqrt{V(f, I)} \leq \frac{1}{2} \left(\frac{V(f, I)}{qS} + qS \right) \leq \frac{1}{2} \left(\frac{\lambda(I)}{q} + qS \right).$$

Using subsequently Lemma 3.3(i), $L \cdot V(f, I) \geq \delta$, $K(f, I) \leq M\lambda(I)$ (see (2.3)) and $i^2 L \leq 2M$, we obtain

$$\begin{aligned}
\sqrt{V(f, I)} &\leq \frac{1}{2} \left(\frac{\lambda(I)}{q} + q(S - i) + q\,i \right) \\
&\leq \frac{1}{2} \left(\frac{\lambda(I)}{q} + qK(f, I) \right) + \frac{1}{2} q \sqrt{V(f, I)}\, \frac{i\sqrt{L}}{\sqrt{\delta}} \\
&\leq \frac{1}{2} \left(\frac{\lambda(I)}{q} + qM\lambda(I) \right) + \frac{1}{2} \left(q \sqrt{V(f, I)}\, \frac{\sqrt{2M}}{\sqrt{\delta}} \right) \\
&= \frac{1}{2} \lambda(I) \sqrt{M} \left(\frac{\sqrt{\delta}}{2} + \frac{2}{\sqrt{\delta}} \right) + \frac{\sqrt{2}}{4} \sqrt{V(f, I)}.
\end{aligned}$$

Substracting the last term on the right hand side from both sides, we obtain the conclusion. $\square$

In the following two definitions, we introduce important technical notions which enable us to formulate, prove, and apply our results in a transparent way.

Definition 3.7. (i) We say that $\mathcal{I} \subset \mathbb{Z}$ is an $\mathbb{Z}$-interval, if $\mathcal{I} = (l, m) \cap \mathbb{Z}$, where $l, m \in \mathbb{Z}^* = \mathbb{Z} \cup \{-\infty, \infty\}$.

(ii) We will say that a family $\mathcal{P}$ of compact intervals is a *generalized partition of a bounded interval* (a, b), if there exists a system $(x_i)_{i \in \mathcal{I}}$ such that $\mathcal{I}$ is an $\mathbb{Z}$-interval, the function $i \mapsto x_i,\ i \in \mathcal{I}$, is strictly increasing, $\inf_{i \in \mathcal{I}} x_i = a$, $\sup_{i \in \mathcal{I}} x_i = b$ and $\mathcal{P} = \{[x_k, x_{k+1}] :\ k, k+1 \in \mathcal{I}\}$.

(iii) We will say that a family $\mathcal{P}$ of compact intervals is a *generalized partition of a bounded open set* $\emptyset \neq G \subset \mathbb{R}$, if $\bigcup\{\mathrm{int}(I) : I \in \mathcal{P}\} \subset G$, and for each component (a, b) of G, the family $\{I \in \mathcal{P} : \mathrm{int}(I) \subset (a, b)\}$ is a generalized partition of (a, b).

(iv) Suppose that $f : [0, 1] \to X$ has locally finite turn in an open set $\emptyset \neq G \subset (0, 1)$, $\mathcal{P}$ is a generalized partition of G, $0 \leq \delta \leq K \leq \infty$, and $\delta \in \mathbb{R}$. Then we say that $\mathcal{P}$ is an $(f, \delta, K; T)$*-partition of* G, if $T(f, I) < \infty$, $T(f, I) \leq K$ for each $I \in \mathcal{P}$, and $T(f, I) \geq \delta$ for each $I \in \mathcal{P}$ with $I \subset G$.

(v) Suppose that $f : [0, 1] \to X$ is continuous and BV. Further suppose that f is not constant on any interval which is a subset of an open set $\emptyset \neq G \subset (0, 1)$ and that the "arc-length parametrization" F of f (see Definition 2.1) is C^2 in $v_f(G)$. Let $\mathcal{P}$ be a generalized partition of G, $0 \leq \delta \leq K \leq \infty$, and $\delta \in \mathbb{R}$. Then we say that $\mathcal{P}$ is an $(f, \delta, K; SV)$*-partition of* G, if $S_I < \infty$, $S_I \cdot V(f, I) \leq K$ for each $I \in \mathcal{P}$, and $S_I \cdot V(f, I) \geq \delta$ for each $I \in \mathcal{P}$ with $I \subset G$, where $S_I = \sup_{x \in v_f(\mathrm{int}(I))} \|F''(x)\|$.

Remark 3.8. Notice that in part (iv) (resp. (v)) of Definition 3.7, if $I = [c, d] \in \mathcal{P}$ but $c \notin G$ or $d \notin G$, then we *do not require* that $T(f, I) \geq \delta$ (resp. $S_I \cdot V(f, I) \geq \delta$).

Definition 3.9. Let $f : [0, 1] \to X$ be continuous and BV. Let $\emptyset \neq G \subset (0, 1)$ be an open set and $0 < \delta < \infty$.

(i) If f has locally finite turn in G, then we define

$$W^\delta(f, G, T) = \sup\Big\{\sum_{k=1}^{n} \sqrt{V(f, I_k)}\Big\},$$

where the supremum is taken over all non-overlapping systems $I_1, \dots, I_n$ of compact intervals with $\mathrm{int}(I_k) \subset G$ such that $T(f, I_k) \geq \delta$ whenever $I_k \subset G$.

(ii) If f is not constant on each interval contained in G and the "arc-length parametrization" F of f (see Definition 2.1) is C^2 on $v_f(G)$, then we define

$$W^\delta(f, G, SV) = \sup\Big\{\sum_{k=1}^{n} \sqrt{V(f, I_k)}\Big\},$$

where the supremum is taken over all non-overlapping systems $I_1, \dots, I_n$ of compact intervals with $\mathrm{int}(I_k) \subset G$ such that $S_{I_k} \cdot V(f, I_k) \geq \delta$ whenever $I_k \subset G$ (where $S_{I_k} = \sup_{x \in v_f(\mathrm{int}(I_k))} \|F''(x)\|$).

Lemma 3.10. *Let $\emptyset \neq G \subset (0, 1)$ be open, and $f : [0, 1] \to X$ be a continuous BV function.*

(i) *Let f have locally finite turn on G and $\delta > 0$. Then there exists a generalized partition $\mathcal{P}$ of G, which is an $(f, \delta, \delta + 3; T)$-partition of G.*

(ii) *Let f be not constant on any subinterval of G, the "arc-length parametrization F" of f be C^2 on $v_f(G)$, and $\delta > 0$. Then there exists a generalized partition $\mathcal{P}$ of G, which is an $(f, \delta, \delta; SV)$-partition of G.*

Proof. Without any loss of generality, we can assume that $G = (a, b) \subset (0, 1)$. Let $x_0 := \frac{a+b}{2}$. For the proof of part (i), we will construct points x_i $(i \in \mathbb{Z})$ with

$$a \leq \cdots \leq x_{-2} \leq x_{-1} < x_0 < x_1 \leq x_2 \leq x_3 \leq \cdots \leq b, \tag{3.2}$$

such that $T_{x_i}^{x_{i+1}} f \le \delta + 3$ if $x_i < x_{i+1}$, and, moreover, with $T_{x_i}^{x_{i+1}} f \ge \delta$ if $x_i < x_{i+1}$, $x_i \ne a$ and $x_{i+1} \ne b$. First, we will construct points $y_n = x_{1-n}$ ($n \in \mathbb{N}$) by induction. So suppose that $n \in \mathbb{N}$ and x_{1-n} was constructed. If $x_{1-n} = a$, then put $x_{-n} := a$. If $x_{1-n} > a$ and $\sup\{T_x^{x_{1-n}} f : x \in (a, x_{1-n})\} \le \delta + 3$, then also put $x_{-n} := a$. Then clearly, $V(\tau_+(f,\cdot), (x_{-n}, x_{1-n})) \le \delta + 3$ and thus $T_{x_{-n}}^{x_{1-n}} f \le \delta + 3$ by Lemma 2.5. If $x_{1-n} > a$ and $\sup\{T_x^{x_{1-n}} f : x \in (a, x_{1-n})\} > \delta + 3$, then set $x_{-n} := \inf\{x \in (a, x_{1-n}) : T_x^{x_{1-n}} f \le \delta + 3\}$. Using Lemma 2.8(iii) (applied to $\mu = x_{1-n}$), we easily obtain $a < x_{-n} < x_{1-n}$. Lemma 2.5 implies $T_{x_{-n}}^{x_{1-n}} f \le \delta + 3$. Moreover, $T_{x_{-n}}^{x_{1-n}} f \ge \delta$. Indeed, otherwise Lemma 2.8(iv) (for $\mu = x_{1-n}$, $x = x_{-n}$) yields a contradiction with the definition of x_{-n}.

We define the points x_n ($n \in \mathbb{N}$) by a symmetrical argument (which is simpler, since we do not need to use Lemma 2.5). It is easy to check that $\mathcal{P} := \{[x_i, x_{i+1}] : i \in \mathbb{Z},\ x_i < x_{i+1}\}$ is a generalized partition of (a, b) with the desired properties.

The proof of part (ii) is quite similar but easier. Indeed, for each $c \in (a, b)$, the function $h(x) := V(f, [c, x]) \cdot \sup_{y \in v_f((c,x))} \|F''(y)\|$ for $x \ne c$, $h(c) := 0$, is clearly continuous on (a, b) and thus we can easily construct points (3.2) such that

$$SV_i := V(f, [x_i, x_{i+1}]) \cdot \sup_{y \in v_f((x_i, x_{i+1}))} \|F''(y)\| \le \delta,$$

and $SV_i = \delta$ whenever $x_i < x_{i+1}$, $x_i \ne a$ and $x_{i+1} \ne b$. It is easy to check that $\mathcal{P} := \{[x_i, x_{i+1}] :\ i \in \mathbb{Z},\ x_i < x_{i+1}\}$ has the desired properties. □

Lemma 3.11. *Let a_i ($i \in I$), b_j, c_j ($j \in J$) be non-negative numbers, I countable, and J finite. Then*

$$\sqrt{\sum_{i \in I} a_i} \le \sum_{i \in I} \sqrt{a_i} \quad \text{and} \quad \sum_{j \in J} \sqrt{b_j c_j} \le \sqrt{\sum_{j \in J} b_j \cdot \sum_{j \in J} c_j}. \tag{3.3}$$

Proof. The first inequality is clear. The second is an immediate consequence of the Cauchy-Schwartz inequality. □

Lemma 3.12. *Let $f : [0,1] \to X$ be continuous and BV, let $\emptyset \ne G \subset (0,1)$ be open, $\delta > 0$, and let $W^\delta(f, G, T)$ (resp. $W^\delta(f, G, SV)$) be defined. Then the following hold.*

(i) *If $\omega : [0,1] \to [0,1]$ is an increasing homeomorphism, then $W^\delta(f, G, T) = W^\delta(f \circ \omega, \omega^{-1}(G), T)$ (resp. $W^\delta(f, G, SV) = W^\delta(f \circ \omega, \omega^{-1}(G), SV)$).*
(ii) *If $\mathcal{P}$ is an $(f, \delta, \infty; T)$-partition of G (resp. $(f, \delta, \infty; SV)$-partition of G), then we have $\sum_{I \in \mathcal{P}} \sqrt{V(f, I)} \le W^\delta(f, G, T)$ (resp. $\sum_{I \in \mathcal{P}} \sqrt{V(f, I)} \le W^\delta(f, G, SV)$).*
(iii) *Let $\mathcal{I}$ be the family of of all components of G. Then $\sum_{I \in \mathcal{I}} \sqrt{V(f, I)} \le W^\delta(f, G, T)$ (resp. $\sum_{I \in \mathcal{I}} \sqrt{V(f, I)} \le W^\delta(f, G, SV)$).*

Proof. Clearly, (2.5) implies (i); part (ii) is obvious. To prove the first inequality of (iii), by Lemma 3.10, choose an $(f, \delta, \infty; T)$-partition $\mathcal{P}$ of G, observe that $\sum_{I \in \mathcal{I}} \sqrt{V(f, I)} \le \sum_{I \in \mathcal{P}} \sqrt{V(f, I)}$ follows from (3.3) and use (ii). The proof of the second inequality of (iii) is quite analogous. □

Lemma 3.13. *Let $f : [0,1] \to X$ be continuous and BV, $\emptyset \ne G \subset (0,1)$ be open, $\delta > 0$, and $W^\delta(f, G, SV)$ be defined. Then*

(i) $W^\delta(f, G, T) \le W^\delta(f, G, SV)$.
(ii) *If $W^\delta(f, G, T) < \infty$, then $\int_{v_f(G)} \sqrt{\|F''\|} < \infty$.*

Proof. By Lemma 2.7 we obtain, for each closed interval $I \subset G$,

$$T(f,I) = \int_{v_f(I)} \|F''\| \leq V(f,I) \cdot \sup_{x \in v_f(\mathrm{int}(I))} \|F''(x)\| < \infty, \tag{3.4}$$

which immediately implies (i).

To prove (ii), suppose that $W^\delta(f,G,T) < \infty$. Choose (by Lemma 3.10(i)) an $(f,\delta,\delta+3;T)$-partition $\mathcal{P}$ of G. Lemma 3.12(ii) implies $\sum_{I\in\mathcal{P}} \sqrt{V(f,I)} < \infty$. By the Cauchy-Schwartz inequality and (3.4) we obtain, for each $I \in \mathcal{P}$ (also for those with $I \setminus G \neq \emptyset$),

$$\begin{aligned}\int_{v_f(I)} 1 \cdot \sqrt{\|F''\|} &\leq \sqrt{\lambda(v_f(I))} \cdot \sqrt{\int_{v_f(I)} \|F''\|} \\ &\leq \sqrt{V(f,I)} \cdot \sqrt{T(f,I)} \leq \sqrt{V(f,I)} \cdot \sqrt{\delta+3}.\end{aligned}$$

Consequently, $\int_{v_f(G)} \sqrt{\|F''\|} \leq \sqrt{\delta+3} \sum_{I\in\mathcal{P}} \sqrt{V(f,I)} < \infty$. □

Lemma 3.14. *Let $f : [0,1] \to X$ have finite convexity, $\emptyset \neq G \subset (0,1)$ be an open set, and let f have locally finite turn on G. Let $f'_+(e) = 0$ or $f'_-(e) = 0$, whenever $e \in (0,1)$ is an endpoint of any component of G. Then $W^\delta(f,G,T) < \infty$ for each $\delta > 0$.*

Proof. Let $I_1,\dots,I_n$ be a system of pairwise non-overlapping compact intervals with $\mathrm{int}(I_k) \subset G$ such that $T(f,I_k) \geq \delta$ whenever $I_k \subset G$. Consider an I_k with $I_k \cap \{0,1\} = \emptyset$. Observing that if $I_k \setminus G \neq \emptyset$, then f'_+ or f'_- vanishes at an endpoint of I_k, by Lemma 3.4, we obtain that $\sqrt{V(f,I_k)} \leq \frac{2+\delta}{2\delta} K^*(f,I_k) + \frac{1}{2}\lambda(I_k)$. So (2.7) implies $\sum_{k=1}^n \sqrt{V(f,I_k)} \leq \frac{2+\delta}{2\delta} K^*(f) + \frac{1}{2} + 2\sqrt{V(f,[0,1])} < \infty$, and thus $W^\delta(f,G,T) < \infty$. □

Lemma 3.15. *Let $f : [0,1] \to X$ be C^2 and let $\emptyset \neq G \subset (0,1)$ be an open set. Let $f'(e) = 0$, whenever $e \in (0,1)$ is an endpoint of any component of G. Let f be non-constant on each interval contained in G and let the "arc-length parametrization" F of f be C^2 on $v_f(G)$. Then $W^\delta(f,G,SV) < \infty$ for each $\delta > 0$.*

Proof. Denote $M^* := \sup_{x\in[0,1]} \|f''(x)\|$, and let $I_1,\dots,I_n$ be a system of non-overlapping compact intervals with $\mathrm{int}(I_k) \subset G$ such that $S_{I_k} \cdot V(f,I_k) \geq \delta > 0$ whenever $I_k \subset G$ (where $S_{I_k} = \sup_{x\in v_f(\mathrm{int}(I_k))} \|F''(x)\|$). If $\inf_{x\in I_k} \|f'(x)\| = 0$, then (2.3) and Lemma 3.4(i) show that $\sqrt{V(f,I_k)} \leq \frac{1}{2}K^*(f,I_k) + \frac{1}{2}\lambda(I_k)$. If $\inf_{x\in I_k} \|f'(x)\| > 0$ and $I_k \subset G$, then Lemma 3.6 implies $\sqrt{V(f,I_k)} \leq \sqrt{M^*}\big(\frac{\sqrt{\delta}}{2} + \frac{2}{\sqrt{\delta}}\big)\lambda(I_k)$. If $\inf_{x\in I_k} \|f'(x)\| > 0$ and $I_k \setminus G \neq \emptyset$, then the assumptions imply $I_k \cap \{0,1\} \neq \emptyset$. Also using (2.7), we obtain

$$\sum_{k=1}^n \sqrt{V(f,I_k)} \leq \sqrt{M^*}\left(\frac{\sqrt{\delta}}{2} + \frac{2}{\sqrt{\delta}}\right) + \frac{K^*(f)+1}{2} + 2\sqrt{V(f,[0,1])} < \infty.$$

Consequently, $W^\delta(f,G,SV) < \infty$. □

Lemma 3.16. *Let a continuous $f : [0,1] \to X$ have bounded variation and let it also have locally bounded turn in an open set $\emptyset \neq G \subset (0,1)$. Suppose that $\mathcal{S}$ is a* *family of pairwise non-overlapping compact intervals such that* $\mathrm{int}(J) \subset G$ *for each* $J \in \mathcal{S}$ *and*

(3.5)
$$\sum_{J\in\mathcal{S}} V(f,J) = V(f,[0,1]),\ \sum_{J\in\mathcal{S}} \sqrt{V(f,J)} < \infty,\ \sum_{J\in\mathcal{S}} \sqrt{V(f,J)\cdot T(f,J)} < \infty.$$

Then $W^{\delta}(f,G,T) < \infty$ *for each* $\delta > 0$.

Proof. Let $\delta > 0$ and consider a finite system $\mathcal{K}$ of non-overlapping compact intervals with $\mathrm{int}(I) \subset G$ $(I \in \mathcal{K})$ such that $T(f,I) \geq \delta$ whenever $I \in \mathcal{K}$ and $I \subset G$. For each $J \in \mathcal{S}$, let $\mathcal{K}_J := \{I \in \mathcal{K} : I \subset \mathrm{int}(J)\}$. Set $\mathcal{K}_1 := \bigcup\{\mathcal{K}_J : J \in \mathcal{S}\}$ and $\mathcal{K}_2 := \mathcal{K} \setminus \mathcal{K}_1$. For each $J \in \mathcal{S}$, we obtain by the second inequality of (3.3) and Lemma 2.6(iv) that

$$\begin{aligned}\sqrt{\delta}\sum\{\sqrt{V(f,I)} : I \in \mathcal{K}_J\} &\leq \sum\{\sqrt{T(f,I)\cdot V(f,I)} : I \in \mathcal{K}_J\}\\ &\leq \sqrt{T(f,J)\cdot V(f,J)}.\end{aligned}$$

Therefore,

$$\sum\{\sqrt{V(f,I)} : I \in \mathcal{K}_1\} \leq (1/\sqrt{\delta})\sum\{\sqrt{T(f,J)\cdot V(f,J)} : J \in \mathcal{S}\}. \tag{3.6}$$

For each $I \in \mathcal{K}_2$, denote by $\mathcal{S}_I$ the set of all $J \in \mathcal{S}$, such that $J \cap \mathrm{int}(I) \neq \emptyset$. If $I = [a,b] \in \mathcal{K}_2$, put $a^* := \min J_a$, if there exists $J_a \in \mathcal{S}$ with $a \in \mathrm{int}(J_a)$ and $a^* := a$, if such J_a does not exist. Similarly, put $b^* := \max J_b$, if there exists $J_b \in \mathcal{S}$ with $b \in \mathrm{int}(J_b)$ and $b^* := b$, if such J_b does not exist. By Lemma 3.2 and the equality of (3.5), we easily obtain that $V(f,[a^*,b^*]) = \sum\{V(f,J) : J \in \mathcal{S}_I\}$. Thus the first inequality of (3.3) implies $\sqrt{V(f,[a^*,b^*])} \leq \sum\{\sqrt{V(f,J)} : J \in \mathcal{S}_I\}$. Observing that, for each $J \in \mathcal{S}$, the set $\{I \in \mathcal{K}_2 : J \in \mathcal{S}_I\}$ contains at most two intervals, we obtain

$$\sum\{\sqrt{V(f,I)} : I \in \mathcal{K}_2\} \leq 2\sum\{\sqrt{V(f,J)} : J \in \mathcal{S}\}. \tag{3.7}$$

Now (3.5), (3.6), and (3.7) imply $W^{\delta}(f,G,T) < \infty$. □

Lemma 3.17. *Let* $f : [0,1] \to X$ *be continuous and BV. Let* $\emptyset \neq G \subset (0,1)$ *be an open set such that* f *is nonconstant on any interval contained in* G *and the "arc-length parametrization"* F *of* f *is* C^2 *on* $v_f(G)$. *Suppose that* $\mathcal{S}$ *is a family of pairwise non-overlapping compact intervals such that* $\mathrm{int}(J) \subset G$ *for each* $J \in \mathcal{S}$ *and*

$$\sum_{J\in\mathcal{S}} V(f,J) = V(f,[0,1]),\ \sum_{J\in\mathcal{S}} \sqrt{V(f,J)} < \infty,\ \sum_{J\in\mathcal{S}} V(f,J)\sqrt{S_J} < \infty, \tag{3.8}$$

where $S_J = \sup_{y\in v_f(\mathrm{int}(J))} \|F''(y)\|$. *Then* $W^{\delta}(f,G,SV) < \infty$ *for each* $\delta > 0$.

Proof. Let $\delta > 0$ and consider a finite system $\mathcal{K}$ of non-overlapping compact intervals such that $\mathrm{int}(I) \subset G$ and $S_I \cdot V(f,I) \geq \delta$ whenever $I \in \mathcal{K}$ and $I \subset G$. For each $J \in \mathcal{S}$, let $\mathcal{K}_J := \{I \in \mathcal{K} : I \subset \mathrm{int}(J)\}$. Set $\mathcal{K}_1 := \bigcup\{\mathcal{K}_J : J \in \mathcal{S}\}$ and $\mathcal{K}_2 := \mathcal{K}\setminus\mathcal{K}_1$. For each $J \in \mathcal{S}$, we obtain

$$\sqrt{\delta}\sum\{\sqrt{V(f,I)} : I \in \mathcal{K}_J\} \leq \sum\{\sqrt{S_I}\cdot V(f,I) : I \in \mathcal{K}_J\} \leq \sqrt{S_J}\cdot V(f,J).$$

Therefore

$$\sum\{\sqrt{V(f,I)} : I \in \mathcal{K}_1\} \leq (1/\sqrt{\delta})\sum\{\sqrt{S_J}\cdot V(f,J) : J \in \mathcal{S}\}. \tag{3.9}$$

In the same way as in the proof of Lemma 3.16 we obtain

$$\sum\{\sqrt{V(f,I)} : I \in \mathcal{K}_2\} \le 2\sum\{\sqrt{V(f,J)} : J \in \mathcal{S}\}. \tag{3.10}$$

Now (3.8), (3.9), and (3.10) imply $W^\delta(f, G, SV) < \infty$. □

In the basic constructions in Sections 5 and 7, we will need the following lemmas.

Lemma 3.18. *Let $(I_\alpha)_{\alpha\in A}$ be a system of pairwise non-overlapping compact subintervals of the interval $[0,d]$. Let $\sum_{\alpha\in A}\mu_\alpha < \infty$, where $\mu_\alpha > 0$, $\alpha \in A$. Then there exists an interval $[0,d']$ and an increasing homeomorphism $\Psi : [0,d'] \to [0,d]$ such that $\lambda(\Psi^{-1}(I_\alpha)) = \mu_\alpha$ and Ψ^{-1} is absolutely continuous.*

Proof. We can define $\Psi := \omega^{-1}$, where $\omega(x) = \int_0^x \varphi$ ($x \in [0,d']$), and $\varphi(t) = \mu_\alpha/\lambda(I_\alpha)$ for $t \in \mathrm{int}(I_\alpha)$ and $\varphi(t) = 1$ for $t \in [0,d'] \setminus \bigcup\{\mathrm{int}(I_\alpha) : \alpha \in A\}$. □

Lemma 3.19. *Let $\mathcal{P}$ be a generalized partition of an open set $\emptyset \ne G \subset (0,d)$. Let $g : [0,d] \to X$ be such that $V := \sum_{I\in\mathcal{P}} V(g, I\cap G) < \infty$, $g(x) = 0$ for each $x \in F := [0,d]\setminus G$, and let g be continuous at all points of F. Then $V(g,[0,d]) < \infty$.*

Proof. Let $\mathcal{J}$ be the family of of all components of G. Using the continuity of g, it is easy to show that, for each $J \in \mathcal{J}$,

$$V(g,\overline{J}) = \sum\{V(g,I) : I \in \mathcal{P}, I \subset \overline{J}\} = \sum\{V(g, I\cap G) : I \in \mathcal{P}, I \subset \overline{J}\}.$$

Now consider arbitrary points $0 = x_0 < x_1 < \dots < x_m = d$. It is easy to see that we can choose points $0 = y_0 < y_1 < \dots < y_n = d$ such that $\{x_0,\dots,x_m\} \subset \{y_0,\dots,y_n\}$ and, for each $0 \le k < n$, either $\{y_k, y_{k+1}\} \subset F$ or $(y_k, y_{k+1}) \subset G$. Then clearly

$$\sum_{i=0}^{m-1}\|g(x_{i+1}) - g(x_i)\| \le \sum_{k=0}^{n-1}\|g(y_{k+1}) - g(y_k)\| \le \sum\{V(g,\overline{J}) : J \in \mathcal{J}\} = V.$$

□

Lemma 3.20. *Let G_1, G_2 be bounded open subsets of $\mathbb{R}$ and $\varphi : G_2 \to G_1$ be an increasing differentiable homeomorphism. Let $Z \subset G_2$ be an interval (of an arbitrary type) such that $V(\varphi', Z) < \infty$, and $h : G_1 \to X$ be such that $V(h'_+, \varphi(Z)) < \infty$. Then*

$$V((h\circ\varphi)'_+, Z) \le \sup_{t\in Z}|\varphi'(t)| \cdot V(h'_+, \varphi(Z)) + \sup_{x\in\varphi(Z)}\|h'_+(x)\| \cdot V(\varphi', Z). \tag{3.11}$$

Proof. Let $t_0 < t_1 < \dots < t_n$ be arbitrary points in Z. Observe that $(h\circ\varphi)'_+(t) = h'_+(\varphi(t))\cdot\varphi'(t)$ for each $t \in Z$, and thus

$$\begin{aligned}
&\|(h\circ\varphi)'_+(t_{i+1}) - (h\circ\varphi)'_+(t_i)\| \\
&\le \|h'_+(\varphi(t_{i+1}))\varphi'(t_{i+1}) - h'_+(\varphi(t_{i+1}))\varphi'(t_i)\| + \|h'_+(\varphi(t_{i+1}))\varphi'(t_i) - h'_+(\varphi(t_i))\varphi'(t_i)\| \\
&\le \sup_{x\in\varphi(Z)}\|h'_+(x)\| \cdot |\varphi'(t_{i+1}) - \varphi'(t_i)| + \sup_{t\in Z}|\varphi'(t)| \cdot \|h'_+(\varphi(t_{i+1})) - h'_+(\varphi(t_i))\|.
\end{aligned}$$

Now the assertion of the lemma follows easily. □

4. The case of real valued functions

As mentioned in Introduction, the functions $f : [0,1] \to \mathbb{R}$ which alow a C^2 parametrization were completely characterized in [LP] and [L] (by conditions (1.2) and (1.1) with $n = 2$). Now, we will show that the same characterization holds, if it is required that the parametrization has finite convexity or a BV derivative. We chose to prove this result easily by using results from [LP] and [L]; however, we could obtain this result also from our theorems on vector functions (see Remark 4.2) which we prove without using results of [LP] and [L]. In Remark 4.3, we note that also the solution of the problem of a C^2 parametrization with almost everywhere nonzero derivatives easily follows from our results. That problem is not considered in [LP] and [L], but the proofs in these articles also give a solution.

Then we present an easy Example 4.4 showing that the "C^2" and "finite convexity" parametrization problems are not equivalent even for vector functions with values in $\mathbb{R}^2$ (see also more sophisticated Examples 6.6, 7.2, 8.4 below).

Theorem 4.1. *Let $f : [0,1] \to \mathbb{R}$ be continuous. Then the following are equivalent.*

(i) *There exists a homeomorphism h of $[0,1]$ onto itself such that $f \circ h$ has finite convexity.*
(ii) *There exists a homeomorphism h of $[0,1]$ onto itself such that $f \circ h$ is $C^{1,BV}$.*
(iii) *There exists a homeomorphism h of $[0,1]$ onto itself such that $f \circ h$ is C^2.*
(iv) *The Lebedev condition (1.1) holds for $n = 2$.*
(v) *The Laczkovich-Preiss condition (1.2) holds for $n = 2$.*

Proof. By results of [LP] and [L] which were mentioned in Introduction, (iv) is equivalent to (v) and implies (iii). Since (iii) implies (ii) and (ii) implies (i) by (2.2), it is sufficient to prove that (i) implies (v). Thus suppose that h as in (i) is given and denote $g := f \circ h$. Let c_i, d_i be as in the definition of $V_{1/2}(g, M_g)$. By Lemma 2.3(iv), we obtain that for each $x \in M_g \cap [0,1)$ we have either $g'_-(x) \le 0 \le g'_+(x)$ or $g'_+(x) \le 0 \le g'_-(x)$ (where $g'_-(0) := 0$). Thus for each $y \in (c_i, d_i)$ we obtain

$$|g'_+(y)| \le |g'_+(c_i) - g'_-(c_i)| + V(g'_+, [c_i, d_i)) =: S_i.$$

Therefore, we subsequently obtain

$$|g(d_i) - g(c_i)| \le S_i\,|d_i - c_i|, \quad |g(d_i) - g(c_i)|^{1/2} \le \sqrt{S_i |d_i - c_i|} \le \frac{S_i + (d_i - c_i)}{2},$$

$$\sum_{i=1}^{m} |g(d_i) - g(c_i)|^{1/2} \le (1/2) \sum_{i=1}^{m} (S_i + (d_i - c_i)) \le C,$$

where C is a constant depending only on g (see (2.7)). Consequently $V_{1/2}(g, M_g) < \infty$, and so also $V_{1/2}(f, M_f) < \infty$, since clearly $M_g = h^{-1}(M_f)$. $\square$

Remark 4.2. We will indicate how the equivalence of (i), (ii), (iii) and (iv) follows from the results of the present article (without using any result of [LP] and [L]). The implications (iii) $\implies$ (ii) $\implies$ (i) are obvious. In the proof of Theorem 4.1 we have proved directly that (i) $\implies$ (v) and thus (i) implies the inequality of (1.1) (for $n = 2$). To prove that (i) implies also the equality of (1.1), let h be as in (i). Lemma 2.3(iii),(iv) and Lemma 3.1 imply $\mathcal{H}^1((f \circ h)(M_{f \circ h})) = 0$; consequently $\mathcal{H}^1(f(M_f)) = 0$. Finally, suppose that (iv) holds. We will show that (iii) holds because f is BV (by Lemma 3.2(i)) and the condition (vii) of

Theorem 7.7 is satisfied. Indeed, let I_α, $\alpha \in A$, be as in (1.1), and let $\mathcal{S}$ be the family of those $\overline{I_\alpha}$ on which f is not constant. Then Lemma 3.2 implies $\sum_{J\in\mathcal{S}} V(f,J) = \sum_{\alpha\in A} V(f,I_\alpha) = V(f,[0,1])$, which is the first equality of (7.18). Since $V(f,I_\alpha) = \omega_\alpha^f$, we obtain from (iv) that f is BV and that the second equality of (7.18) holds. Since f is strictly monotone on each $J \in \mathcal{S}$, we have that F is linear on $v_f(J)$. So $\mathcal{S}_J = 0$ and the third equality of (7.18) holds.

Finally, note that Lemma 3.2(i),(ii) easily implies that the Laczkovich-Preiss condition (1.2) is equivalent to the Lebedev condition (1.1) (for any $n \in \mathbb{N}$) but the precise proof is not short.

Remark 4.3. Remark 7.9, Theorem 7.7, and Theorem 4.1 immediately imply that, for a continuous $f : [0,1] \to \mathbb{R}$, the following are equivalent:

(a) *There exists a homeomorphism h of $[0,1]$ onto itself such that $f\circ h$ is C^2 and $(f\circ h)'(x) \neq 0$ almost everywhere.*
(b) *f is nonconstant on any interval and (1.1) or (1.2) holds for $n = 2$.*

Example 4.4. There exists a function $f : [0,1] \to \mathbb{R}^2$ with finite convexity which does not admit a C^1 parametrization.

Proof. Take $\{a_n : n \in \mathbb{N}\}$ to be a dense subset of $(0,1)$, and $g : [0,1] \to \mathbb{R}$ a convex Lipschitz function with $g'_+(a_n) \neq g'_-(a_n)$ for all n. Let $f(x) = (x, g(x))$, $x \in [0,1]$. Clearly, f has finite convexity. Suppose that f admits a C^1 parametrization. Then there exists an increasing homeomorphism of $[0,1]$ onto itself such that $f \circ h$ is C^1 on $[0,1]$. Clearly h is C^1 on $[0,1]$ and thus there exists an open interval $I \subset [0,1]$ such that $h' > 0$ on I and thus h^{-1} is differentiable on $h(I)$. Choose $j \in \mathbb{N}$ with $a_j \in h(I)$. Since $g = (g\circ h)\circ h^{-1}$, we obtain that g is differentiable at a_j, a contradiction. □

5. Parametrizations with finite convexity

The problem of a parametrization with finite convexity and non-zero unilateral derivatives has a very simple solution.

Proposition 5.1. *Let X be a Banach space, and let $f : [0,1] \to X$ be continuous. There exists a homeomorphism h of $[0,1]$ onto itself such that $f\circ h$ has finite convexity, $(f\circ h)'_+(x) \neq 0$ for $x \in [0,1)$, and $(f\circ h)'_-(x) \neq 0$ for $x \in (0,1]$, if and only if $T_0^1 f < \infty$.*

Proof. Let h be the homeomorphism with the required properties. Then [D1, Proposition 5.11] and (2.5) easily imply that f has finite turn. (Alternatively, we can apply Lemma 3.3(iii) to $f\circ h$, since Lemma 2.3(ii) easily implies that $\inf_{x\in[0,1)} \|(f\circ h)'_+(x)\| > 0$.)

Suppose that $T_0^1 f < \infty$. Then Lemma 2.6(ii) implies that $F = f \circ v_f^{-1}$ has finite convexity and $F'_\pm(x) = \tau_\pm(F,x) \neq 0$ for all $x \in [0,\ell)$ (resp. $x \in (0,\ell]$, where $\ell = v_f(1)$). Thus we can put $h(t) := v_f^{-1}(\ell \cdot t)$ for $t \in [0,1]$. □

Lemma 5.2. *Suppose that a continuous $f : [0,1] \to X$ has bounded variation, $\emptyset \neq G \subset (0,1)$ is an open set and f has locally finite turn in G. Suppose that $\mathcal{P}$ is an $(f,0,K;T)$-partition of G for some $0 < K < \infty$ such that $\sum_{I\in\mathcal{P}} \sqrt{V(f,I)} < \infty$ and $\mathcal{H}^1(f(H)) = 0$, where $H := [0,1] \setminus G$. Then there exists an increasing homeomorphism $h : [0,1] \to [0,1]$ such that:*

(i) *$f \circ h$ has finite convexity and $(f \circ h)'_+(x) \neq 0$, $(f \circ h)'_-(x) \neq 0$ for each $x \in h^{-1}(G)$;*
(ii) *if f is tangentially smooth on G, then $f \circ h$ is C^1;*
(iii) *if f is nonconstant on any interval, then $\lambda(h^{-1}(H)) = 0$.*

Proof. Let U be the maximal open set on which f is locally constant. Set $v^*(x) := v_f(x) + \lambda([0, x] \cap U)$, $x \in [0, 1]$. It is clear that v^* is continuous and increasing. Put $d_1 := v^*(1)$ and $\xi := (v^*)^{-1}$; clearly $\xi : [0, d_1] \to [0, 1]$ is an increasing homeomorphism. Denote $f_1 := f \circ \xi$, $G_1 := \xi^{-1}(G)$ and $\mathcal{P}_1 := \{\xi^{-1}(I) : I \in \mathcal{P}\}$. Clearly $\sum\{\sqrt{\lambda(J)} : J \in \mathcal{P}_1\} < \infty$. By (2.5) we have that $T(f_1, J) \leq K$ for each $J \in \mathcal{P}_1$.

If the interval (γ, δ) is a component of G_1, then $f_1|_{[\gamma,\delta]}$ is clearly an arc-length parametrization of $f|_{[\xi(\gamma),\xi(\delta)]}$. So Lemma 2.6(i) implies that

$$\|(f_1)'_+(x)\| = \|(f_1)'_-(x)\| = 1 \quad \text{whenever } x \in G_1, \text{ and} \tag{5.1}$$

$$f_1 \text{ is } C^1 \text{ on } G_1 \text{ if } f \text{ is tangentially smooth on } G. \tag{5.2}$$

Using (5.1), Lemma 2.6(ii) and Lemma 2.3(v), we easily get

$$V((f_1)'_+, J \cap G_1) \leq K + 2 \quad \text{whenever } J \in \mathcal{P}_1. \tag{5.3}$$

If $J = [\alpha, \beta] \in \mathcal{P}_1$, set $c(J) := \sqrt{\lambda(J)}$. Further, if $\alpha \notin G$, let $b(J) := 0$, and if $\alpha \in G$, set $b(J) := \min(c(J), c(J_l))$, where $J_l \in \mathcal{P}_1$ is the interval, whose right endpoint is α. Similarly, if $\beta \notin G$, let $d(J) := 0$, and if $\beta \in G$, set $d(J) := \min(c(J), c(J_r))$, where $J_r \in \mathcal{P}_1$ is the interval, whose left endpoint is β. Finally set $\mu(J) := 6\lambda(J)(b(J) + 4c(J) + d(J))^{-1}$.

Since $\mu(J) \leq 2\sqrt{\lambda(J)}$, we can use Lemma 3.18 and choose $0 < d_2 < \infty$ and an increasing homeomorphism $\Psi : [0, d_2] \to [0, d_1]$ such that $\lambda(\Psi^{-1}(J)) = \mu(J)$ for each $J \in \mathcal{P}_1$. Set $\mathcal{P}_2 := \{\Psi^{-1}(J) : J \in \mathcal{P}_1\}$, $G_2 := \Psi^{-1}(G_1)$ and $H_2 := [0, d_2] \setminus G_2$.

For each interval $I = [p, q] \in \mathcal{P}_2$, let $J := \Psi(I)$ and g_I be the continuous function on I such that $g_I(p) = b(J)$, $g_I(q) = d(J)$, $g_I(x) = c(J)$ for each $x \in [p + (q - p)/3, p + 2(q - p)/3]$, and g_I is linear on the intervals $[p, p + (q - p)/3]$, $[p + 2(q - p)/3, q]$. Further put $\varphi_I(x) := \Psi(p) + \int_p^x g_I$ for $x \in I$. We see that

$$\varphi_I(q) - \varphi_I(p) = (q - p) \cdot (b(J) + 4c(J) + d(J))/6 = \lambda(J).$$

Thus, setting $\varphi(x) := \varphi_I(x)$ if $x \in I \in \mathcal{P}_2$ and $\varphi(x) := \Psi(x)$ if $x \in [0, d_2] \setminus \bigcup \mathcal{P}_2$, we easily see that $\varphi : [0, d_2] \to [0, d_1]$ is an increasing homeomorphism which is C^1 with $\varphi' > 0$ on G_2 and $\varphi'(x) \leq c(J)$ whenever $x \in I \cap G_2$, where $I \in \mathcal{P}_2$ and $J = \varphi(I)$.

Put $f_2 := f_1 \circ \varphi$. Consider $I \in \mathcal{P}_2$ and $J = \varphi(I) \in \mathcal{P}_1$. Using (5.1), we obtain that

$$\|(f_2)'_-(x)\| = \|(f_2)'_+(x)\| = \|\varphi'(x)\| = \|g_I(x)\| \leq c(J) \quad \text{if } \ x \in I \cap G_2. \tag{5.4}$$

Using (5.3), (5.1), and the obvious inequality $V(\varphi', I \cap G_2) \leq 2c(J)$, we apply Lemma 3.20 (with $Z := I \cap G_2$ and $h := f_1$) and obtain that

$$V((f_2)'_+, I \cap G_2) \leq (K + 2)c(J) + 2c(J),$$

and therefore

$$\sum_{I \in \mathcal{P}_2} V((f_2)'_+, I \cap G_2) < \infty. \tag{5.5}$$

Now we will show that (setting $(f_2)'_+(d_2) := 0$)

$$\text{for each } u \in H_2,\ (f_2)'(u) = 0 \text{ and } (f_2)'_+ \text{ is continuous at } u. \tag{5.6}$$

Let $u \in H_2$ and $\varepsilon > 0$ be given. If u is a left endpoint of some $I \in \mathcal{P}_2$, observe that $g_I(u) = 0$ and g_I is right continuous at u. Thus (5.4) and [B, Chap. I, par. 2, Proposition 3] clearly imply that there exists $v > u$ such that f_2 is ε-Lipschitz on $[u, v]$.

If $u \neq d_2$ and u is not a left endpoint of an $I \in \mathcal{P}_2$, then observe that $c(J) > \varepsilon$ for finitely many $J \in \mathcal{P}_1$ and therefore we can by (5.4) find a $v > u$ such that $\|(f_2)'_+(x)\| \leq \varepsilon$ for each $x \in G_2 \cap [u, v]$. So [B, Chap. I, par. 2, Proposition 3] implies that f_2 is ε-Lipschitz on each component of $G_2 \cap (u, v)$. Since clearly $\mathcal{H}^1(f_2(H_2)) = 0$, Lemma 3.2(iii) implies that f_2 is ε-Lipschitz on $[u, v]$.

Quite similarly we get for each $0 \neq u \in H_2$ a $v < u$ such that f_2 is ε-Lipschitz on $[v, u]$. Now it is easy to see that (5.6) holds. Using (5.5) and (5.6), we apply Lemma 3.19 (with $G := G_2$ and $g := (f_2)'_+$, $g(d_2) = 0$) and obtain, also using (2.2), that g has finite convexity on $[0, d_2]$.

By (5.4) and the definition of g_I we have that $(f_2)'_+(x) \neq 0$ and $(f_2)'_-(x) \neq 0$ for each $x \in G_2$. Moreover, if f is tangentially smooth on G, then (5.2) implies that f_2 is C^1 on G_2. Using (5.6), we obtain that f_2 is C^1 on $[0, d_2]$.

Thus, to finish the proof of (i) and (ii), it is sufficient to define $h := \xi \circ \varphi \circ \pi$, where $\pi(x) = d_2 x,\ x \in [0, 1]$. To prove (iii), suppose that f is nonconstant on any interval. Then $v^* = v_f = \xi^{-1}$; so Lemma 3.2(iv) implies $\lambda(\xi^{-1}(H)) = 0$. Since Ψ^{-1} is absolutely continuous (see Lemma 3.18) and $\varphi^{-1}(\xi^{-1}(H)) = \Psi^{-1}(\xi^{-1}(H))$, we have $\lambda((\xi \circ \varphi)^{-1}(H)) = 0$, and thus also $\lambda(h^{-1}(H)) = 0$. □

Let $f : [0, 1] \to X$. We define the set T_f as the set of all points in $[0, 1]$ such that there is no open interval U containing x such that f has finite turn on $\overline{U}$. Clearly, T_f is closed, $\{0, 1\} \subset T_f$, and f is not constant on any interval $I \subset [0, 1] \setminus T_f$. Further, if h is a homeomorphism of $[0, 1]$ onto itself, then (2.5) implies

$$T_{f \circ h} = h^{-1}(T_f). \tag{5.7}$$

We need the following simple lemma.

Lemma 5.3. *Suppose that $f : [0, 1] \to X$ has finite convexity, and $x \in T_f$. Then we have that either $x \in \{0, 1\}$ or $f'_+(x) = 0$ or $f'_-(x) = 0$.*

Proof. Let $x \in (0, 1) \cap T_f$ with $f'_+(x) \neq 0$ and $f'_-(x) \neq 0$. By Lemma 2.3(ii) there exists $\delta > 0, \eta > 0$ such that $\|f'_+(x)\| \geq \eta$ for all $y \in [x - \delta, x + \delta]$. Thus Lemma 3.3(iii) implies that

$$T(f, [x - \delta, x + \delta]) \leq \frac{2}{\eta} K(f, [x - \delta, x + \delta]) < \infty,$$

and we have a contradiction with $x \in T_f$. □

The following is the main characterization theorem of curves allowing a parametrization with finite convexity (observe that (i) below clearly implies that f is continuous and BV).

Theorem 5.4. *Let $f : [0, 1] \to X$ be BV continuous nonconstant and $G := [0, 1] \setminus T_f$. Then the following are equivalent.*

(i) *There exists a homeomorphism h of $[0, 1]$ onto itself such that $f \circ h$ has finite convexity.*

(ii) *There exists a homeomorphism* φ *of* $[0,1]$ *onto itself such that* $f \circ \varphi$ *has finite convexity, and* $(f \circ \varphi)'_{\pm}(x) \neq 0$ *for each* $x \in \varphi^{-1}(G)$.
(iii) $\mathcal{H}^1(f(T_f)) = 0$ *and* $W^\delta(f, G, T) < \infty$ *for each* $\delta > 0$.
(iv) $\mathcal{H}^1(f(T_f)) = 0$ *and* $W^\delta(f, G, T) < \infty$ *for some* $\delta > 0$.
(v) $\mathcal{H}^1(f(T_f)) = 0$ *and* $\sum_{I \in \mathcal{P}} \sqrt{V(f, I)} < \infty$ *whenever* $\mathcal{P}$ *is an* $(f, \delta, \infty; T)$*-partition of* G *with* $\delta > 0$.
(vi) $\mathcal{H}^1(f(T_f)) = 0$ *and* $\sum_{I \in \mathcal{P}^*} \sqrt{V(f, I)} < \infty$ *for some* $(f, 0, \widetilde{K}; T)$*-partition* $\mathcal{P}^*$ *of* G *with* $\widetilde{K} < \infty$.
(vii) *There exists a family* $\mathcal{S}$ *of pairwise non-overlapping compact intervals such that* $\operatorname{int}(J) \subset G$ *for each* $J \in \mathcal{S}$ *and*

$$\sum_{J \in \mathcal{S}} V(f, J) = V(f, [0,1]), \ \sum_{J \in \mathcal{S}} \sqrt{V(f, J)} < \infty, \ \sum_{J \in \mathcal{S}} \sqrt{V(f, J) \cdot T(f, J)} < \infty. \tag{5.8}$$

If f *is nonconstant on any interval, then in (ii) we can also assert* $\lambda(\varphi^{-1}(G)) = 1$.

Proof. First note that (since f is nonconstant) both $\mathcal{H}^1(f(T_f)) = 0$ and the equality of (5.8) clearly imply $G \neq \emptyset$. The implications (ii) $\Longrightarrow$ (i) and (iii) $\Longrightarrow$ (iv) are trivial; (v) $\Longrightarrow$ (vi) holds on account of Lemma 3.10(i). Lemma 3.12(ii) implies (iii) $\Longrightarrow$ (v) and (vii) $\Longrightarrow$ (iii) by Lemma 3.2(ii) and Lemma 3.16. If (vi) holds, then Lemma 5.2 gives (ii) (and also $\lambda(\varphi^{-1}(G)) = 1$ if f is nonconstant on any interval).

To show that (iv) $\Longrightarrow$ (vi), let $\delta > 0$ be as in (iv), and put $\widetilde{K} := \delta + 3$. Lemma 3.10(i) implies that there exists an $(f, \delta, \widetilde{K}; T)$-partition $\mathcal{P}^*$ of G and we have that $\sum_{I \in \mathcal{P}^*} \sqrt{V(f, I)} < \infty$ by Lemma 3.12(ii).

To prove (vi) $\Longrightarrow$ (vii) suppose that (vi) holds and $\mathcal{P}^*$ is given. Put $\mathcal{S} := \mathcal{P}^*$. Then the equality of (5.8) holds by Lemma 3.2(i) (applied to $G := \bigcup_{P \in \mathcal{S}} \operatorname{int}(P)$). Since $\sqrt{T(f, J)} \leq \sqrt{\widetilde{K}}$ for each $J \in \mathcal{S}$, also both inequalities of (5.8) hold.

To prove that (i) $\Longrightarrow$ (iii), let h be as in (i), put $g := f \circ h$ and $G^* := h^{-1}(G)$. Using (5.7), Lemma 5.3 and Lemma 3.1, we easily obtain $\mathcal{H}^1(f(T_f)) = 0$. By (5.7), Lemma 5.3, and Lemma 3.14 we obtain $W^\delta(g, G^*, T) < \infty$; so $W^\delta(f, G, T) < \infty$ by Lemma 3.12(i). □

Remark 5.5. (i) The conditions (v) and (vi) give an "algorithmic" way how to decide whether (i) holds:

Decide whether $\mathcal{H}^1(f(T_f)) = 0$. If it holds, then choose an $(f, \delta, K; T)$-partition $\mathcal{P}$ of $G := [0,1] \setminus T_f$ with $\delta > 0$ and $K < \infty$ (such a partition exists by Lemma 3.10(i)) and decide whether $\sum_{I \in \mathcal{P}} \sqrt{V(f, I)} < \infty$.

(ii) The condition (vii) is elegant and needs no auxiliary notions for its formulation. On the other hand, it is not easily applicable in the case when (i) does not hold.
(iii) For a very simple necessary and sufficient condition in an interesting, very special case see Proposition 7.11.

Remark 5.6. Let $f : [0,1] \to X$ be BV continuous and $G := [0,1] \setminus T_f$. Then the following are equivalent.

(a) *There exists a homeomorphism* h *of* $[0,1]$ *onto itself such that* $f \circ h$ *has finite convexity and* $(f \circ h)'_{\pm}(x) \neq 0$ *almost everywhere.*
(b) f *is nonconstant on any interval and any of conditions (iii)–(vii) holds.*

It follows immediately from Theorem 5.4 and the simple observation that (a) implies that f is nonconstant on any interval.

Proposition 5.7. *Let a nonconstant function $f : [0,1] \to X$ allow an equivalent parametrization with finite convexity. Set $G := [0,1] \setminus T_f$. Suppose that the "arc-length parametrization" F of f (see Definition 2.1) is C^2 on $v_f(G)$. Then $\int_{v_f(G)} \sqrt{\|F''\|} < \infty$.*

Proof. For each $\delta > 0$, we have $W^\delta(f, G, T) < \infty$ by Theorem 5.4. Thus the assertion follows from Lemma 3.13. □

Concerning natural questions about the strength of the condition from the preceding proposition, see Example 8.3, and Proposition 7.11.

6. Smooth parametrizations with finite convexity

The problem of a C^1 parametrization with finite convexity and non-zero derivatives has a very simple solution.

Proposition 6.1. *Let X be a Banach space, and let $f : [0,1] \to X$ be continuous. There exists a homeomorphism h of $[0,1]$ onto itself such that $f \circ h$ is a C^1 function which has finite convexity, and $(f \circ h)'(x) \neq 0$ for $x \in [0,1]$ if and only if $\tau(f,x)$ exists for all $x \in [0,1]$ and $T_0^1 f < \infty$.*

Proof. Let h be a homeomorphism with the required properties. Since $(f \circ h)'(x) \neq 0$, we have that $\tau(f \circ h, x)$ exists for each $x \in [0,1]$, and thus $\tau(f,y)$ exists for each $y \in [0,1]$. Proposition 5.1 gives $T_0^1 f < \infty$.

Suppose that $T_0^1 f < \infty$, and that $\tau(f,x)$ exists for each $x \in [0,1]$. Then Lemma 2.6(ii) implies that F has finite convexity, and Lemma 2.6(i) implies that $F'(x) = \tau(F,x) \neq 0$ at all points $x \in [0,\ell]$ (where $\ell = v_f(1)$); F is C^1 by Lemma 2.3(ii). Thus we can put $h(t) := v_f^{-1}(\ell \cdot t)$, $t \in [0,1]$. □

Note that the necessary and sufficient condition from the above proposition can be written (by Lemma 2.6(v)) in an alternative form: *f has finite turn and is tangentially smooth.*

For $f : [0,1] \to X$ we define S_f as the set of all points $x \in [0,1]$ such that there is no neighbourhood U of x such that f is either constant or tangentially smooth on U. Clearly S_f is closed and $\{0,1\} \subset S_f$. Further, if h is a homeomorphism of $[0,1]$ onto itself, then clearly

$$S_{f\circ h} = h^{-1}(S_f). \tag{6.1}$$

The following theorem was proved in [DZ].

Theorem 6.2 ([DZ])**.** *Let X be a Banach space, and let $f : [0,1] \to X$. Then there is a homeomorphism h of $[0,1]$ onto itself such that $f \circ h$ is C^1 if and only if f is continuous, BV, and $\mathcal{H}^1(f(S_f)) = 0$.*

Lemma 6.3. *Let $g : [0,1] \to X$ be C^1 and $x \in S_g$. Then either $g'(x) = 0$ or $x \in \{0,1\}$.*

Proof. Suppose that $x \in (0,1) \cap S_g$. For a contradiction, assume that $g'(x) \neq 0$. Since g is C^1, there exist $\eta, \delta > 0$ such that $\|g'(y)\| > \eta$ for all $y \in [x-\delta, x+\delta]$. But then $\tau(g,y) = g'(y)/\|g'(y)\|$ is continuous on $[x-\delta, x+\delta]$, and we have a contradiction with $x \in S_g$. □

The main result of the present section is the following.

Theorem 6.4. *Let $f : [0,1] \to X$ be BV continuous nonconstant and $G := [0,1] \setminus (T_f \cup S_f)$. Then the following are equivalent.*

(i) *There exists a homeomorphism h of $[0,1]$ onto itself such that $f \circ h$ is a C^1 function with finite convexity.*
(ii) *There exists a homeomorphism φ of $[0,1]$ onto itself such that $f \circ \varphi$ is C^1 with finite convexity, and $(f \circ \varphi)'(x) \neq 0$ for each $x \in \varphi^{-1}(G)$.*
(iii) *$\mathcal{H}^1(f(T_f \cup S_f)) = 0$ and $W^\delta(f, G, T) < \infty$ for each $\delta > 0$.*
(iv) *$\mathcal{H}^1(f(T_f \cup S_f)) = 0$ and $W^\delta(f, G, T) < \infty$ for some $\delta > 0$.*
(v) *$\mathcal{H}^1(f(T_f \cup S_f)) = 0$ and $\sum_{I \in \mathcal{P}} \sqrt{V(f, I)} < \infty$ for each $\mathcal{P}$ which is an $(f, \delta, \infty; T)$-partition of G with $\delta > 0$.*
(vi) *$\mathcal{H}^1(f(T_f \cup S_f)) = 0$ and $\sum_{I \in \mathcal{P}^*} \sqrt{V(f, I)} < \infty$ for some $(f, 0, \widetilde{K}; T)$-partition $\mathcal{P}^*$ of G with $\widetilde{K} < \infty$.*
(vii) *There exists a family $\mathcal{S}$ of pairwise non-overlapping compact intervals such that $\mathrm{int}(J) \subset G$ for each $J \in \mathcal{S}$ and*

$$\sum_{J \in \mathcal{S}} V(f, J) = V(f, [0,1]), \ \sum_{J \in \mathcal{S}} \sqrt{V(f, J)} < \infty, \ \sum_{J \in \mathcal{S}} \sqrt{V(f, J) \cdot T(f, J)} < \infty. \tag{6.2}$$

If f is nonconstant on any interval, then in (ii) we can also assert $\lambda(\varphi^{-1}(G)) = 1$.

Proof. The proof is literally the same as the proof of Theorem 5.4, except the implication (i) $\Longrightarrow$ (iii).

To prove that (i) $\Longrightarrow$ (iii), let h be as in (i), put $g := f \circ h$ and $G^* = h^{-1}(G)$. Using (5.7), (6.1), Lemma 5.3, Lemma 6.3, and Lemma 3.1, we easily obtain $\mathcal{H}^1(f(T_f \cup S_f)) = 0$. By (5.7), (6.1), Lemma 5.3, Lemma 6.3, and Lemma 3.14 we obtain $W^\delta(g, G^*, T) < \infty$; so $W^\delta(f, G, T) < \infty$ by Lemma 3.12(i). $\square$

Remark 6.5. Let $f : [0,1] \to X$ be BV continuous and $G := [0,1] \setminus (T_f \cup S_f)$. Then the following are equivalent (cf. Remark 5.6).

(a) *There exists a homeomorphism h of $[0,1]$ onto itself such that $f \circ h$ is C^1, has finite convexity and $(f \circ h)'(x) \neq 0$ for a.e. $x \in [0,1]$.*
(b) *f is nonconstant on any interval and any of conditions (iii)–(vii) holds.*

The following example suggests that Theorem 6.4 cannot be easily deduced from Theorem 5.4.

Example 6.6. There exists a function $f : [0,1] \to \mathbb{R}^2$ with finite convexity, such that $f \circ h$ is a C^1 function for some homeomorphism h of $[0,1]$ onto itself, but $f \circ \varphi$ is not a C^1 function with finite convexity for any homeomorphism φ of $[0,1]$ onto itself.

Proof. Find $1 > x_1 > x_2 > \cdots > 0$ such that $\lim x_n = 0$ and $x_j - x_{j+1} = cj^{-2}$ for some $c > 0$. Choose a Lipschitz convex function g on $[0,1]$ for which $\{x_1, x_2, ..\}$ is the set of all points $x \in (0,1)$ at which g is not differentiable. Let $f(x) := (x, g(x))$, $x \in [0,1]$. It is easy to see that f has finite convexity, $S_f = \{0, 1, x_1, x_2, \dots\}$, and $T_f = \{0, 1\}$. Therefore, there exists a homeomorphism h of $[0,1]$ onto itself such that $f \circ h$ is a C^1 function by Theorem 6.2. On the other hand, since $\sum_{j=1}^\infty \sqrt{V(f, [x_{j+1}, x_j])} \geq \sum_{j=1}^\infty \sqrt{cj^{-2}} = \infty$, $f \circ \varphi$ is not a C^1 function with finite convexity for any homeomorphism φ of $[0,1]$ onto itself. Indeed, otherwise the condition (iii) of Theorem 6.4 holds and so Lemma 3.12(iii) gives a contradiction. □

Remark 6.7. Example 6.6 shows that the problems of a parametrization with finite convexity and of a $C^{1,BV}$ parametrization are not necessarily equivalent even when $T_f = \{0, 1\}$ and S_f is countable with only one point of accumulation. On the other hand, these problems are equivalent when $S_f \setminus T_f$ is finite. Indeed, given a family $\mathcal{S}$ from the condition (vii) of Theorem 5.4, we can obtain a family $\mathcal{S}^*$ verifying (vii) of Theorem 6.4 by dividing finitely many members of $\mathcal{S}$ into a finite number of intervals.

7. C^2-parametrizations

We say that a Banach space X *has a Fréchet smooth norm*, provided the norm $\|\cdot\|$ of X is Fréchet differentiable at all $x \in X \setminus \{0\}$ (or equivalently $\|\cdot\|$ is C^1 on $X \setminus \{0\}$). The following proposition shows that, for such X, the characterization of those $f : [0, 1] \to X$, which allow a C^2 parametrization with nonzero derivatives, is quite simple.

Proposition 7.1. *Let X be a Banach space with a Fréchet smooth norm, let $f : [0, 1] \to X$ be continuous. Then the following are equivalent.*

(i) *There exists a homeomorphism h of $[0, 1]$ onto itself such that $f \circ h$ is C^2 with $(f \circ h)'(x) \neq 0$ for all $x \in [0, 1]$.*
(ii) *f is BV, is not constant on any interval, and $F := f \circ v_f^{-1}$ is C^2.*

Proof. Let h be as in (i). We can suppose that h is increasing. Let $g := f \circ h$. Note that $v_g(x) = \int_0^x \|g'(t)\|\, dt$ is C^2, with $(v_g)'(x) = \|g'(x)\| > 0$, for $x \in [0, 1]$, and thus v_g^{-1} is C^2 on $v_g([0, 1])$. It follows that $F = g \circ v_g^{-1}$ is C^2.

Now suppose that (ii) holds. Set $\ell := v_f(1)$ and $h(t) := v_f^{-1}(\ell \cdot t)$, $t \in [0, 1]$. Using Lemma 2.7 and Lemma 2.6(i), we see that h has the property from (i). □

The following example is based on a simple "smoothness argument". For a more interesting Example 8.4 we need our main results.

Example 7.2. There exists a $C^{1,BV}$ function $f : [0, 1] \to \mathbb{R}^2$ such that for any homeomorphism h of $[0, 1]$ onto itself we have that $f \circ h$ is not C^2.

Proof. Let $g : [0, 1] \to \mathbb{R}$ be a continuous BV function such that g is not C^1 on any subinterval of $[0, 1]$. Define $G(x) := \int_0^x g(t)\, dt$ and $f(x) := \big(x, G(x)\big)$ for $x \in [0, 1]$. It is easy to see that f is $C^{1,BV}$. On the other hand, suppose that h is an increasing homeomorphism of $[0, 1]$ onto itself such that $f \circ h$ is C^2. Then h is C^2. Find an open interval $I \subset [0, 1]$ and $\zeta > 0$ such that $h'(x) > \zeta$ for each $x \in I$. Then h^{-1} is C^2 on $J := h(I)$. Therefore $G = (G \circ h) \circ h^{-1}$ is C^2 on J, a contradiction. □

Let $f : [0, 1] \to X$, and let D_f be the set of all points $x \in [0, 1]$ for which there is no interval $(c, d) \subset [0, 1]$ containing x such that $f|_{[c,d]}$ has a C^2 arc-length parametrization. The set D_f is obviously closed and $\{0, 1\} \subset D_f$. Further, if h is a homeomorphism of $[0, 1]$ onto itself, then clearly

$$D_{f \circ h} = h^{-1}(D_f). \tag{7.1}$$

Lemma 7.3. *Let X be a Banach space with a Fréchet smooth norm, let $g : [0, 1] \to X$ be C^2, and let $x \in D_g$. Then either $g'(x) = 0$ or $x \in \{0, 1\}$.*

Proof. Suppose that $x \in (0,1) \cap D_g$. For a contradiction, assume $g'(x) \neq 0$. Then there exists $\delta > 0$ such that $g'(y) \neq 0$ for each $y \in [x-\delta, x+\delta]$. Now observe that the proof of Proposition 7.1 (with $[x-\delta, x+\delta]$ instead of $[0,1]$) gives that $g|_{[x-\delta,x+\delta]}$ has a C^2 arc-length parametrization, which contradicts $x \in D_g$. $\square$

Lemma 7.4. *Let $0 < d < 1$, $\xi > 0$ and $c_l, c_r \geq 0$ with $0 < \max(c_l, c_r) \leq 10^{-9} \cdot \xi d$ be given. Then, for each $I = [u,v]$ with $\lambda(I) = d$, there exists a C^2 function ω on I such that $\omega(u) = \omega(v) = 0$, $\omega'(x) = c_l$ for $x \in [u, u+d/3]$, $\omega'(x) = c_r$ for $x \in [u+2d/3, v]$, and $\max(|\omega'(x)|, |\omega''(x)|) \leq \xi$ for $x \in I$.*

Proof. We can suppose that $I = [0,d]$. First assume that $c_l \leq c_r$. Denote $p_1 := 2(9c_r - c_l)\xi^{-1}$, $p_2 := 20 \cdot c_r \cdot \xi^{-1}$; clearly

$$0 < p_1 \leq p_2 < 10^{-7} d. \tag{7.2}$$

For every $0 \leq s \leq d/3 - p_1 - p_2$, let $x_0 := 0$, $x_1 := d/3$, $x_2 := x_1 + p_1/2$, $x_3 := x_2 + p_1/2$, $x_4 := x_3 + s$, $x_5 := x_4 + p_2/2$, $x_6 := x_5 + p_2/2$, $x_7 := d$. Clearly $x_i \leq x_{i+1}$ $(0 \leq i \leq 6)$. Now consider the function $\mu = \mu_s$ which is linear on each $[x_i, x_{i+1}]$ (if $x_i < x_{i+1}$) and

$$\mu(x_0) = \mu(x_1) = \mu(x_3) = \mu(x_4) = \mu(x_6) = \mu(x_7) = 0,\ \mu(x_2) = -\xi,\ \mu(x_5) = \xi.$$

Clearly $|\mu(x)| \leq \xi$, $x \in [0,d]$. Let $\nu(x) = \nu_s(x) = c_l + \int_0^x \mu$, $x \in I$. It is easy to see that $\nu(x) = c_l$ for $x \in [x_0, x_1]$, $\nu(x) = -9c_r$ for $x \in [x_3, x_4]$, $\nu(x) = c_r$ for $x \in [x_6, x_7]$, and $|\nu(x)| \leq 9c_r \leq \xi$ for $x \in I$. These properties of ν and (7.2) easily imply that $g(s) := \int_0^d \nu_s > 0$ for $s = 0$ and $g(s) < 0$ for $s = d/3 - p_1 - p_2$. Since $g(s)$ is clearly continuous, we can choose $s_0 \in (0, d/3 - p_1 - p_2)$ with $g(s_0) = 0$. Now it is easy to see that $\omega(x) := \int_0^x \nu_{s_0}$, $x \in I$, has all desired properties.

If $c_l > c_r$, we apply the just-proven assertion to $I^* := [-d, 0]$, $c_l^* := c_r$, $c_r^* := c_l$ and obtain a function ω^* on I^*. Now it is sufficient to put $\omega(x) := -\omega^*(-x)$, $x \in [0,d]$. $\square$

Lemma 7.5. *Let $V > 0$, $\eta > 0$ and $c_l, c_r \geq 0$ such that $d := \sqrt{V/\eta} < 1$ and $\max(c_l, c_r) \leq 10^{-10} \cdot d^2\eta$ be given. Then, for every interval J of the length V and every interval $I = [u,v]$ of the length d, there exists an increasing C^2 homeomorphism $\varphi : I \to J$ such that $\varphi'(u) = c_l$, $\varphi'(v) = c_r$, $\varphi''(u) = \varphi''(v) = 0$, $0 < \varphi'(x) \leq 19 \cdot \sqrt{\eta V}$ for $x \in \operatorname{int}(I)$, and $|\varphi''(x)| \leq 19 \cdot \eta$ for $x \in I$.*

Proof. We can suppose that $J = [0,V]$ and $I = [0,d]$. Consider the function π on I which is linear on each $[id/6, (i+1)d/6]$ $(i = 0, \dots, 5)$, $\pi(d/6) = \eta$, $\pi(5d/6) = -\eta$ and $\pi(id/6) = 0$ for $i = 0, 2, 3, 4, 6$. Define $\rho(x) := \int_0^x \pi$, $x \in I$, and $\tau(x) := \int_0^x \rho$, $x \in I$. Clearly $\rho(0) = \rho(d) = 0$, $0 < \rho(x) \leq \eta d$ for $x \in (0,d)$ and $\rho(x) = d\eta/6$ for $x \in [d/3, 2d/3]$. Therefore $d^2\eta/18 \leq \tau(d) \leq \eta d^2$. Thus, setting $\Psi(x) = (\eta d^2/\tau(d))\tau(x)$ $(x \in I)$, we have $\Psi(d) = \eta d^2 = V$. Consequently, $\Psi : I \to J$ is an increasing homeomorphism with $\Psi' > 0$ on $\operatorname{int}(I)$. Further

$$|\Psi''(x)| \leq 18 \cdot |\tau''(x)| \leq 18 \cdot \eta, \quad \Psi'(x) \leq 18 \cdot \tau'(x) \leq 18 \cdot \eta d = 18\sqrt{\eta V} \quad (x \in I), \tag{7.3}$$

$$\Psi'(0) = \Psi'(d) = 0, \quad \text{and} \quad \Psi'(x) \geq \tau'(x) = \rho(x) = \frac{d\eta}{6}, \quad x \in \left[\frac{d}{3}, \frac{2d}{3}\right]. \tag{7.4}$$

Therefore, in the case $c_l = c_r = 0$, the function $\varphi := \Psi$ has the desired properties.

If $\max(c_l, c_r) > 0$, set $\xi := d\eta/7$. Then $\max(c_l, c_r) \leq 10^{-10} \cdot d^2\eta \leq 10^{-9} \cdot \xi d$. Thus we can use Lemma 7.4 and obtain a corresponding function ω on $[0,d]$. Define $\varphi :=$

$\Psi + \omega$. Since $|\omega'(x)| \leq d\eta/7$ if $x \in [d/3, 2d/3]$ and $\omega'(x) \geq 0$ otherwise, using (7.4) we obtain that $\varphi' > 0$ on $\mathrm{int}(I)$. Since $\omega(0) = \omega(d) = 0$, we obtain that φ is a homeomorphism of I onto J. Since $\omega'(0) = c_l$, $\omega'(d) = c_r$, $\max(|\omega'(x)|, |\omega''(x)|) \leq \xi$ for $x \in I$, and $\xi = \sqrt{V\eta}/7 = d\eta/7 < \eta$, we obtain by (7.3) and (7.4) that φ has all desired properties. □

Lemma 7.6. *Suppose that $f : [0,1] \to X$ is BV continuous, $\emptyset \neq G \subset (0,1)$ is an open set, f is not constant on any subinterval of G and (the "arc-length parametrization", see Definition 2.1) F is C^2 on $v_f(G)$. Let $\mathcal{H}^1(f(H)) = 0$, where $H := [0,1] \setminus G$. Suppose that $0 < K < \infty$ and $\mathcal{P}$ is an $(f, 0, K; SV)$-partition of G such that $\sum_{I\in\mathcal{P}} \sqrt{V(f,I)} < \infty$. Then there exists a homeomorphism $h : [0,1] \to [0,1]$ such that $f \circ h$ is C^2 and $(f \circ h)'(x) \neq 0$ for each $x \in h^{-1}(G)$.*

Moreover, if f is nonconstant on any interval, then $\lambda(h^{-1}(H)) = 0$.

Proof. Let U be the maximal open set on which f is locally constant. Set $v^*(x) := v_f(x) + \lambda([0,x] \cap U)$, $x \in [0,1]$. It is clear that v^* is continuous and increasing. Put $d_1 := v^*(1)$ and $\xi := (v^*)^{-1}$; clearly $\xi : [0, d_1] \to [0,1]$ is an increasing homeomorphism. Denote $f_1 := f \circ \xi$, $G_1 := \xi^{-1}(G)$ and $\mathcal{P}_1 := \{\xi^{-1}(I) : I \in \mathcal{P}\}$. Clearly $\sum\{\sqrt{\lambda(J)} : J \in \mathcal{P}_1\} < \infty$.

If (γ, δ) is a component of G_1, then $f_1|_{[\gamma,\delta]}$ is clearly an arc-length parametrization of $f|_{[\xi(\gamma),\xi(\delta)]}$. Consequently, f_1 is C^2 on G_1,

$$\|f_1'(x)\| = 1 \quad \text{for} \quad x \in G_1, \quad \text{and} \tag{7.5}$$

$$\sup_{s\in\mathrm{int}(J)} \|f_1''(s)\| \cdot \lambda(J) \leq K \qquad \text{for each} \quad J \in \mathcal{P}_1. \tag{7.6}$$

It is easy to see that to each $J \in \mathcal{P}_1$ we can assign a number $\eta_J > 0$ such that

$$\sum_{J\in\mathcal{P}_1} \sqrt{\lambda(J)/\eta_J} < \infty, \qquad \sqrt{\lambda(J)/\eta_J} < 1 \ (J \in \mathcal{P}_1) \quad \text{and} \tag{7.7}$$

$$\{J \in \mathcal{P}_1 : \ \eta_J > \varepsilon\} \quad \text{is finite for each} \quad \varepsilon > 0. \tag{7.8}$$

Denote $d_J := \sqrt{\lambda(J)/\eta_J}$. For each point x which is an endpoint of a member of $\mathcal{P}_1$, choose $c(x) \geq 0$ such that

$$c(x) = 0 \quad \text{whenever} \quad x \notin G_1 \quad \text{and} \tag{7.9}$$

$$0 < c(x) \leq 10^{-10} \cdot (d_J)^2 \eta_J \quad \text{if} \quad x \in G_1 \quad \text{is an endpoint of} \quad J \in \mathcal{P}_1. \tag{7.10}$$

Since $\sum_{J\in\mathcal{P}_1} d_J < \infty$, by Lemma 3.18 we can choose $0 < d_2 < \infty$ and an increasing homeomorphism $\Psi : [0, d_2] \to [0, d_1]$ such that $\lambda(\Psi^{-1}(J)) = d_J$ for each $J \in \mathcal{P}_1$. Set $\mathcal{P}_2 := \{\Psi^{-1}(J) : \ J \in \mathcal{P}_1\}$, $G_2 := \Psi^{-1}(G_1)$ and $H_2 := [0, d_2] \setminus G_2$.

Now consider an interval $I = [u, v] \in \mathcal{P}_2$. Let $J := \Psi(I)$, $\eta := \eta_J$, $d := d_J$, $V := \lambda(J)$, $c_l(I) = c_l := c(\Psi(u))$, and $c_r(I) = c_r := c(\Psi(v))$.

Since $\max(c_l, c_r) \leq 10^{-10} \cdot d^2\eta$ by (7.10) and $d < 1$ by (7.7), we can choose by Lemma 7.5 an increasing C^2 homeomorphism $\varphi_I : I \to J$ such that

$$\varphi_I'(u) = c_l, \ \varphi_I'(v) = c_r, \ 0 < \varphi_I'(x) \leq 19 \cdot \sqrt{\eta V} \quad (x \in \mathrm{int}(I)), \tag{7.11}$$

$$\varphi_I''(u) = \varphi_I''(v) = 0 \quad \text{and} \quad |\varphi_I''(x)| \leq 19 \cdot \eta \quad (x \in I). \tag{7.12}$$

Thus, setting $\varphi(x) := \varphi_I(x)$ if $x \in I \in \mathcal{P}_2$ and $\varphi(x) := \Psi(x)$ if $x \in [0, d_2] \setminus \bigcup \mathcal{P}_2$, we easily see that $\varphi : [0, d_2] \to [0, d_1]$ is an increasing homeomorphism. Further, the definition of $c_l(I)$, $c_r(I)$, (7.11) and (7.12) easily imply that φ is C^2 on G_2

and $\varphi' > 0$ on G_2. Consequently, also $f_1 \circ \varphi$ is C^2 on G_2. By (7.5), we have $(f_1 \circ \varphi)'(x) \neq 0$ for $x \in G_2$. Now consider $I = \Psi^{-1}(J) \in \mathcal{P}_2$ and $x \in I \cap G_2$. Then

$$(7.13)\qquad (f_1 \circ \varphi)''(x) = (f_1 \circ \varphi_I)''(x) = f_1''(\varphi_I(x)) \cdot (\varphi_I'(x))^2 + f_1'(\varphi_I(x)) \cdot \varphi_I''(x).$$

Consequently, (recall that $\eta = \eta_J$ and $V = \lambda(J)$) by (7.5), (7.6), (7.11), (7.12), and (7.7) we have

$$(7.14)\qquad \|(f_1 \circ \varphi)''(x)\| \leq \frac{K}{V} \cdot 19^2 \cdot \eta_J V + 19 \cdot \eta_J \leq \eta_J (19^2 \cdot K + 19) \quad \text{and}$$

$$(7.15)\qquad \|(f_1 \circ \varphi)'(x)\| = \|f_1'(\varphi_I(x)) \cdot \varphi_I'(x)\| \leq 19 \cdot \sqrt{\eta_J \lambda(J)}.$$

To prove that $f_1 \circ \varphi$ is C^2 on $[0, d_2]$, it is sufficient to show that

$$(7.16)\qquad \text{for each } \ z \in H_2,\ (f_1 \circ \varphi)''(z) = 0 \text{ and } (f_1 \circ \varphi)'' \text{ is continuous at } z.$$

First, we will prove that $(f_1 \circ \varphi)'_+(z) = 0$ for each $z \in H_2 \setminus \{d_2\}$. To this end we consider an arbitrary $\varepsilon > 0$. We will show that there exists a $\delta > 0$ such that $\|(f_1 \circ \varphi)'(x)\| < \varepsilon$ for each $x \in (z, z + \delta) \cap G_2$. Indeed, if z is not the left endpoint of some $I \in \mathcal{P}_2$, the existence of such a δ follows from (7.8) and (7.15). If z is the left endpoint of some $I \in \mathcal{P}_2$, we find a desired δ easily from the first equality of (7.15), (7.5), (7.9) and the first equality of (7.11). Since clearly $\mathcal{H}^1((f_1 \circ \varphi)(H_2)) = 0$, using the mean value theorem and Lemma 3.2(iii), we easily obtain that $f_1 \circ \varphi$ is ε-Lipschitz on $[z, z + \delta]$. Thus $(f_1 \circ \varphi)'_+(z) = 0$. Similarly we obtain that $(f_1 \circ \varphi)'_-(z) = 0$ for each $z \in H_2 \setminus \{0\}$.

To prove (7.16), consider $\varepsilon > 0$ and $z \in H_2 \setminus \{d_2\}$. Now observe that there exists a $\delta > 0$ such that

$$(7.17)\qquad \|(f_1 \circ \varphi)''(x)\| < \varepsilon \quad \text{for each } \ x \in (z, z + \delta) \cap G_2.$$

Indeed, if z is not the left endpoint of some $I \in \mathcal{P}_2$, then (7.17) follows from (7.8) and (7.14). If z is the left endpoint of some $I \in \mathcal{P}_2$, we obtain (7.17) easily from (7.13), (7.5), (7.6), (7.9), and the equalities of (7.11) and (7.12). Now consider an arbitrary $y \in (z, z + \delta) \cap G_2$; let w be the left endpoint of the component of G_2 which contains y. Then $(f_1 \circ \varphi)'(w) = 0$ and thus the mean value theorem and (7.17) imply $\|(f_1 \circ \varphi)'(y)\| \leq \varepsilon|y - w| \leq \varepsilon|y - z|$. Since $(f_1 \circ \varphi)'(y) = 0$ for each $y \in (z, z + \delta) \setminus G_2$, we see that $\lim_{y \to z+} (f_1 \circ \varphi)'(y)/|y - z| = 0$. Also using a symmetrical argument, we conclude that $(f_1 \circ \varphi)''(z) = 0$ for each $z \in [0, d_2] \setminus G_2$. The rest of (7.16) now easily follows from (7.17) and its symmetrical version. Thus, to finish the proof of the first part of the assertion, it is sufficient to define $h := \xi \circ \varphi \circ \pi$, where $\pi(x) = d_2 x,\ x \in [0, 1]$.

The proof of the second part of the assertion is quite analogous to the end of the proof of Lemma 5.2 (we use that $\xi^{-1} = v_f$ and Ψ^{-1} is absolutely continuous). □

The main result of the present section is the following theorem (observe that condition (i) implies that f is continuous and BV).

Theorem 7.7. *Let X be a Banach space with a Fréchet smooth norm. Let a BV continuous nonconstant $f : [0, 1] \to X$ be given, and put $G := [0, 1] \setminus D_f$. Then the following are equivalent.*

- (i) *There exists a homeomorphism h of $[0, 1]$ onto itself such that $f \circ h$ is a C^2 function.*
- (ii) *There exists a homeomorphism φ of $[0, 1]$ onto itself such that $f \circ \varphi$ is C^2 and $(f \circ \varphi)'(x) \neq 0$ for each $x \in \varphi^{-1}(G)$.*

(iii) $\mathcal{H}^1(f(D_f)) = 0$ *and* $W^\delta(f, G, SV) < \infty$ *for each* $\delta > 0$.
(iv) $\mathcal{H}^1(f(D_f)) = 0$ *and* $W^\delta(f, G, SV) < \infty$ *for some* $\delta > 0$.
(v) $\mathcal{H}^1(f(D_f)) = 0$ *and* $\sum_{I\in\mathcal{P}} \sqrt{V(f,I)} < \infty$ *whenever* $\mathcal{P}$ *is an* $(f, \delta, \infty; SV)$*-partition of* G *with* $\delta > 0$.
(vi) $\mathcal{H}^1(f(D_f)) = 0$ *and* $\sum_{I\in\mathcal{P}^*} \sqrt{V(f,I)} < \infty$ *for some* $(f, 0, K^*; SV)$*-partition* $\mathcal{P}^*$ *of* G *with* $K^* < \infty$.
(vii) *There exists a family* $\mathcal{S}$ *of pairwise non-overlapping compact intervals such that* $\operatorname{int}(J) \subset G$ *for each* $J \in \mathcal{S}$ *and*

$$\sum_{J\in\mathcal{S}} V(f,J) = V(f,[0,1]), \quad \sum_{J\in\mathcal{S}} \sqrt{V(f,J)} < \infty, \quad \sum_{J\in\mathcal{S}} V(f,J)\cdot\sqrt{S_J} < \infty, \tag{7.18}$$

where $S_J = \sup_{y\in v_f(\operatorname{int}(J))} \|F''(y)\|$.

If f *is nonconstant on any interval, then in (ii) we can also assert* $\lambda(\varphi^{-1}(G)) = 1$.

Proof. Note that (since f is nonconstant) both $\mathcal{H}^1(f(D_f)) = 0$ and the equality of (7.18) clearly imply $G \neq \emptyset$. The implications (ii) $\Longrightarrow$ (i) and (iii) $\Longrightarrow$ (iv) are trivial; (v) $\Longrightarrow$ (vi) holds on account of Lemma 3.10(ii). Lemma 3.12(ii) implies (iii) $\Longrightarrow$ (v) and (vii) $\Longrightarrow$ (iii) by Lemma 3.2(ii) and Lemma 3.17. If (vi) holds, then Lemma 7.6 gives (ii) (and also $\lambda(\varphi^{-1}(G)) = 1$ if f is nonconstant on any interval).

To show that (iv) $\Longrightarrow$ (vi), let $\delta > 0$ be as in condition (iv), and put $K^* := \delta$. Lemma 3.10(ii) now implies that there exists an $(f, \delta, K^*; SV)$-partition $\mathcal{P}^*$ of G and we have that $\sum_{I\in\mathcal{P}^*} \sqrt{V(f,I)} < \infty$ by Lemma 3.12(ii).

To prove (vi) $\Longrightarrow$ (vii) suppose that (vi) holds and $\mathcal{P}^*$ is given. Put $\mathcal{S} := \mathcal{P}^*$. Then the equality of (7.18) holds by Lemma 3.2(i) (applied to $G := \bigcup_{P\in\mathcal{S}} \operatorname{int}(P)$). Since $V(f,J)\cdot\sqrt{S_J} \leq \sqrt{K^*}\cdot\sqrt{V(f,J)}$ for each $J \in \mathcal{S}$, also both inequalities of (7.18) hold.

To prove that (i) $\Longrightarrow$ (iii), let h be as in (i), put $g := f \circ h$ and $G^* = h^{-1}(G)$. Using (7.1), Lemma 7.3, and Lemma 3.1, we easily obtain $\mathcal{H}^1(f(D_f)) = 0$.

Let $\delta > 0$. By (7.1), Lemma 7.3 and Lemma 3.15, we obtain $W^\delta(g, G^*, SV) < \infty$; so $W^\delta(f, G, SV) < \infty$ by Lemma 3.12(i). $\square$

Remark 7.8. (i) For the proof of (iii)-(vii) implies (i) or (ii) in Theorem 7.7, no assumption concerning the smoothness of X was needed.
(ii) Note (cf. the analogous more detailed Remark 5.5) that the conditions (v) and (vi) give an "algorithmic" way how to decide whether (i) holds. The condition (vii) is elegant and needs no auxiliary notions for its formulation, but it is not easily applicable if (i) does not hold. A very simple necessary condition is given in Proposition 7.10. In Proposition 7.11, we show that this condition is also sufficient in an interesting, very special case.

Remark 7.9. Let X be a Banach space with a Fréchet smooth norm. Let $f : [0,1] \to X$ be BV continuous and $G := [0,1] \setminus D_f$. Then the following are equivalent.

(a) *There exists a homeomorphism* h *of* $[0,1]$ *onto itself such that* $f \circ h$ *is* C^2 *and* $(f \circ h)'(x) \neq 0$ *almost everywhere.*
(b) f *is nonconstant on any interval and any of conditions* $(iii) - (vii)$ *holds.*

It follows immediately from Theorem 7.7 and the simple observation that (a) implies that f is nonconstant on any interval.

Proposition 7.10. *Suppose that X has a Fréchet smooth norm. Let a nonconstant $f:[0,1]\to X$ allow an equivalent C^2 parametrization. Set $G:=[0,1]\setminus D_f$. Then $\int_{v_f(G)}\sqrt{\|F''\|}<\infty$.*

Proof. By Theorem 7.7, for each $\delta>0$, we have $W^\delta(f,G,SV)<\infty$. Thus the assertion follows from Lemma 3.13(i),(ii). □

Proposition 7.11. *Let X be a Banach space with a Fréchet smooth norm. Assume that $f:[0,1]\to X$ is continuous, BV and nonconstant on any interval. Let $F:=f\circ v_f^{-1}$ and $\ell:=v_f(1)$. Suppose that F'' is continuous on $(0,\ell)$ and for some $\delta>0$ we have that $\|F''\|$ is monotone on $(0,\delta)$ and on $(\ell-\delta,\ell)$.*

Then the following are equivalent.

(i) $\int_0^\ell\sqrt{\|F''(t)\|}\,dt<\infty$,
(ii) *There exists a homeomorphism h of $[0,1]$ onto itself such that $f\circ h$ is C^2.*
(iii) *There exists a homeomorphism φ of $[0,1]$ onto itself such that $f\circ\varphi$ has finite convexity.*

Proof. To prove that (i) $\Longrightarrow$ (ii), choose points $(x_i)_{i\in\mathbb{Z}}$ in $(0,\ell)$ such that $\mathcal{P}:=\{[x_i,x_{i+1}]:\ i\in\mathbb{Z}\}$ is a generalized partition of $(0,\ell)$ and $x_{i+1}-x_i\sim i^{-4},\ |i|\to\infty$. For sufficiently large $|i|$, clearly

$$\int_{x_i}^{x_{i+1}}\sqrt{\|F''(t)\|}\,dt\ge(x_{i+1}-x_i)\min(\sqrt{\|F''(x_i)\|},\sqrt{\|F''(x_{i+1})\|}).$$

Thus, using $i^{-4}\sim(i+1)^{-4}$, we easily obtain $\sum_{i\in\mathbb{Z}\setminus\{0\}}\sqrt{\|F''(x_i)\|}\,i^{-4}<\infty$. Put $\mathcal{S}:=\{v_f^{-1}(I):\ I\in\mathcal{P}\}$. Once more, using the assumption concerning the monotonicity of $\|F''\|$, it is now easy to check that (7.18) from Theorem 7.7 holds, which implies (ii).

The implication (ii) $\Longrightarrow$ (iii) follows from (2.3) and (iii) $\Longrightarrow$ (i) from Proposition 5.7. □

Example 8.3 shows that the assumption on the monotonicity of $\|F''\|$ cannot be omitted in Proposition 7.11.

Now we will state a simple formulation of a consequence of our results, which might be interesting from the point of view of differential geometry. Recall that "regular", here, means "C^1 with nonzero derivative".

Proposition 7.12. *Let $a,b\in\mathbb{R}$ and $f:(a,b)\to\mathbb{R}^n$ be a C^2 curve parametrized by the arc-length. Let $\kappa_1(s)=\|f''(s)\|$, $s\in(a,b)$, be the first curvature of f. Let $0<\delta\le K<\infty$ and $\mathcal{P}$ be a generalized partition of (a,b) (see Definition 3.7(ii)) such that $\mu(I):=\lambda(I)\cdot\sup_{s\in\operatorname{int}(I)}\kappa_1(s)\le K$ for all $I\in\mathcal{P}$, and $\mu(I)\ge\delta$ for $I\in\mathcal{P}$ with $I\subset(a,b)$ (such $\mathcal{P}$ exists; see the proof of Lemma 3.10(ii)).*

Then the following are equivalent:

(i) *f has a regular C^2 parametrization $f^*:(c,d)\to\mathbb{R}^n$, $c,d\in\mathbb{R}$, such that $(f^*)''$ is bounded.*
(ii) *There exists a C^2 function $g:[u,v]\to\mathbb{R}^n$ such that $g|_{(u,v)}$ is a regular parametrization of f.*
(iii) $\sum_{I\in\mathcal{P}}\sqrt{\lambda(I)}<\infty$.

In the case when κ_1 is monotone on some right neigbourhood of a and on some left neighbourhood of b these conditions are equivalent to

(iv) $\int_a^b \sqrt{\kappa_1(s)}\, ds < \infty$.

Proof. Since f is 1-Lipschitz, there exists a continuous extension $\hat{f} : [a,b] \to X$ of f. It is easy to see that any g from (ii) is a parametrization of $\hat{f}$ and thus (ii) is equivalent to (iii) by Theorem 7.7 (the equivalence of (ii), (v) and (vi)). The implication (ii) $\Longrightarrow$ (i) is obvious.

To show that (i) $\Longrightarrow$ (ii), let f^* be as in (i). The mean value theorem implies that $(f^*)'$ is bounded, consequently f^* is Lipschitz and thus f^* extends to a Lipschitz function $\tilde{f}$ on $[c,d]$. Let $\omega : [c,d] \to [c,d]$ be an increasing C^2 homeomorphism such that $\omega'(c) = \omega''(c) = \omega'(d) = \omega''(d) = 0$ and $\omega'(y) \neq 0$ for all $y \in (c,d)$. Such an ω obviously exists. Put $g := \tilde{f} \circ \omega$, $u := c$, $v := d$. Then g is C^2 on (u,v) and $g'(t) = (\tilde{f})'(\omega(t))\, \omega'(t)$ for $t \in (u,v)$. Therefore $g|_{(u,v)}$ is regular and, since $(\tilde{f})'$ is bounded on (u,v), we have $\lim_{t\to u+} g'(t) = 0$. Thus the mean value theorem easily gives that g is C^1 on $[u,v)$ with $g'(u) = 0$. Since $g''(t) = (\tilde{f})''(\omega(t))\cdot(\omega'(t))^2 + (\tilde{f})'(\omega(t))\cdot\omega''(t)$ for $t \in (u,v)$ and $(\tilde{f})'$, $(\tilde{f})''$ are bounded on (u,v), we obtain $\lim_{t\to u+}(g')'(t) = 0$ which implies that g' is C^1 on $[u,v)$ with $(g')'(u) = 0$. The situation at v is symmetric.

Finally, assume that κ_1 is monotone on $(a, a+\delta)$ and $(b-\delta, b)$ for some $\delta > 0$. Since any g from (ii) is a parametrization of $\hat{f}$, the equivalence of (ii) and (iv) follows easily from Proposition 7.11 and Theorem 7.7 (the equivalence of (i) and (ii)). $\square$

8. Examples

The following example shows an application of Theorem 5.4.

Example 8.1. For $s > 0$, consider the spiral $f : [0,1] \to \mathbb{R}^2$ defined by $f(0) = 0$ and

$$f(t) = (x(t), y(t)) = (t^s \cos(1/t), t^s \sin(1/t)), \quad 0 < t \leq 1.$$

Using the well-known formula for the oriented curvature $k(t)$ (see e.g. [S, p. 26]) we obtain

$$k(t) = \frac{\det(f'(t), f''(t))}{\|f'(t)\|^3} = \frac{s(1-s)t^{2s-4} - t^{2s-6}}{t^{3s-6}(1+s^2t^2)^{3/2}} < 0, \quad t > 0.$$

Since $k(t) < 0$ (i.e. the tangent vector is “rotating to the right”), it is geometrically obvious (and easy to prove) that $\mathcal{P} := \{[(2(k+1)\pi)^{-1}, (2k\pi)^{-1}], k \in \mathbb{N}\}$ is an $(f, \pi, 4\pi; T)$ partition of $(0, (2\pi)^{-1})$. Further it is geometrically obvious (because of $k(t) < 0$) and easy to prove that $V(f, [(2(k+1)\pi)^{-1}, (2k\pi)^{-1}]) \asymp k^{-s}$. Thus we see that f is BV if and only if $s > 1$ and Theorem 5.4 easily implies that f has a parametrization with finite convexity if and only if $\sum_{k\in\mathbb{N}} \sqrt{k^{-s}} < \infty$, which holds if and only if $s > 2$. In this case f has also a C^2 parametrization. It follows from Proposition 7.11, since it is easy to check that $k(t)$ (and thus also $\|F''(s)\|$) is monotone on a right neighbourhood of 0. Of course, we could also check the condition (i) from Proposition 7.11, but it would need additional computations.

In the following two examples, we need the following well-known fact.

Lemma 8.2. *Let $k : (0,1) \to \mathbb{R}$ be positive and C^∞. Then there is a continuous $f : [0,1] \to \mathbb{R}^2$ parametrized by the arc-length, C^∞ on $(0,1)$, and such that $\|f''(x)\| = k(x)$ for $x \in (0,1)$.*

Proof. By the Fundamental Theorem of the local theory of curves (see e.g. [K, Theorem 2.15]), there exists $g : (0,1) \to \mathbb{R}^2$ parametrized by the arc-length, which is C^∞, and $\|g''(x)\| = k(x)$ for $x \in (0,1)$. Since g is 1-Lipschitz, it has a continuous extension f to $[0,1]$, which has all the desired properties. $\square$

Example 8.3. There exists a continuous $f : [0,1] \to \mathbb{R}^2$ which is parametrized by the arc-length, is C^∞ on $(0,1)$, does not allow a parametrization with finite convexity, but $\int_0^1 \sqrt{\|f''\|} < \infty$.

Proof. Let $\mathcal{P} = \{I_n : n \in \mathbb{N}\}$ be a generalized partition of $(0,1)$ such that $\lambda(I_n) = c/n^2$ for some $c > 0$. Choose closed intervals $J_n \subset I_n$ with $\lambda(J_n) = c/n^4$. We can clearly choose a positive C^∞ function $k : (0,1) \to \mathbb{R}$ such that, for each $n \in \mathbb{N}$, we have $\max_{x \in J_n} k(x) = n^4$, $k(x) \le 1$ for $x \in I_n \setminus J_n$, and $\int_{J_n} k \ge c/2$. Choose f corresponding to k by Lemma 8.2. By Lemma 2.7 we see that $\mathcal{P}$ is an $(f, c/2, \infty; T)$-partition of $(0,1)$, but $\sum_{I \in \mathcal{P}} \sqrt{\lambda(I)} = \infty$, and therefore f does not allow a parametrization with finite convexity (see Theorem 5.4(v)). On the other hand, $\int_0^1 \sqrt{\|f''\|} \le 1 + \sum (c/n^4)\sqrt{n^4} < \infty$. $\square$

Example 8.4. There exists a continuous $f : [0,1] \to \mathbb{R}^2$ parametrized by the arc-length such that f is C^∞ on $(0,1)$, $f \circ h$ is $C^{1,BV}$ for some homeomorphism h of $[0,1]$ onto itself, but $f \circ \varphi$ is not a C^2 function on $[0,1]$ for any homeomorphism φ of $[0,1]$ onto itself.

Proof. Let $\mathcal{P} = \{I_n : n \in \mathbb{N}\}$ be a generalized partition of $(0,1)$ such that $\lambda(I_n) = c/n^4$ for some $c > 0$. Divide each interval I_n into closed subintervals J_n^j ($j = 1, \dots, j_n$) so that $\sum_j \sqrt{\lambda(J_n^j)} \ge 1$ and put $\mathcal{P}^* := \{J_n^j : n \in \mathbb{N}, 1 \le j \le j_n\}$. We can clearly choose a positive C^∞ function $k : (0,1) \to \mathbb{R}$ such that $\max_{x \in J} k(x) = 1/\lambda(J)$ for each $J \in \mathcal{P}^*$ and $\int_I k \le 1$ for each $I \in \mathcal{P}$. Choose f corresponding to k by Lemma 8.2. By Lemma 2.7, $\mathcal{P}$ is an $(f, 0, 1; T)$-partition of $(0,1)$ with $\sum_{I \in \mathcal{P}} \sqrt{\lambda(I)} < \infty$, and thus there exists a homeomorphism h of $[0,1]$ onto itself such that $f \circ h$ is $C^{1,BV}$ by Theorem 6.4. On the other hand, $\mathcal{P}^*$ is an $(f, 1, \infty; SV)$-partition of $(0,1)$, and, since $\sum_{J \in \mathcal{P}^*} \sqrt{\lambda(J)} = \infty$, we obtain (see Theorem 7.7(v)) that $f \circ \varphi$ is not a C^2 function on $[0,1]$ for any homeomorphism φ of $[0,1]$ onto itself. $\square$

Acknowledgment

The research of the first author was supported in part by ISF. The research of the second author was supported by the institutional grant MSM 0021620839 and the grants GAČR 201/03/0931 and GAČR 201/06/0198. Both authors would also like to thank for hospitality to Erwin Schrödinger Institute, Vienna, where part of the work was done.

References

[AR] A. D. Alexandrov, Yu. G. Reshetnyak, *General theory of irregular curves*, Mathematics and its Applications (Soviet Series), 29, Kluwer Academic Publishers Group, Dordrecht, 1989.

[B] N. Bourbaki, *Éléments de mathématique. I: Les structures fondamentales de l'analyse. Livre IV: Fonctions d'une variable réelle (théorie élémentaire),* Second ed., Act. Sci. et Ind. vol. 1074, Hermann, Paris, 1958.

[BL] Y. Benyamini, J. Lindenstrauss, *Geometric Nonlinear Functional Analysis, Vol. 1,* Colloquium Publications **48**, American Mathematical Society, Providence, 2000.
[B] A. Bruckner, *Differentiation of real functions*, Second edition, CRM Monograph Series **5**, American Mathematical Society, Providence, 1994.
[BG] A. M. Bruckner, C. Goffman, *Differentiability through change of variables,* Proc. Amer. Math. Soc. **61** (1976), 235–241.
[C] G. Choquet, *Application des propriétés descriptives de la fonction contingent à la théorie des fonctions de variable réelle et à la géométrie différentielle des variétés cartésiennes*, J. Math. Pures Appl. (9) **26** (1947), 115–226 (1948).
[D1] J. Duda, *Curves with finite turn*, to appear in the Czechslovak Math. J.[1]
[D2] J. Duda, *On positively differentiable curves with values in normed linear spaces*, to appear in J. Math. Anal. Appl.
[DZ] J. Duda, L. Zajíček, *Curves in Banach spaces - differentiability via homeomorphisms*, to appear in Rocky Mountain J. Math.[1]
[F] H. Federer, *Geometric Measure Theory,* Grundlehren der math. Wiss., vol. 153, Springer, New York, 1969.
[GNW] C. Goffman, T. Nishiura, D. Waterman, *Homeomorphisms in analysis*, Mathematical Surveys and Monographs, vol. 54, AMS, Providence, RI, 1997.
[K] B. Kirchheim, *Rectifiable metric spaces: local structure and regularity of the Hausdorff measure*, Proc. Amer. Math. Soc. **121** (1994), 113–123.
[K] W. Kühnel, *Differential geometry, Curves–surfaces–manifolds*, Student Mathematical Library, 16, American Mathematical Society, Providence, RI, 2002.
[LP] M. Laczkovich, D. Preiss, *α-variation and transformation into C^n functions*, Indiana Univ. Math. J. **34** (1985), 405–424.
[L] V. V. Lebedev, *Homeomorphisms of a segment and smoothness of a function* (Russian), Mat. Zametki **40** (1986), 364–373, 429.
[MS] J. L. Massera, J. J. Schäffer, *Linear differential equations and functional analysis, I*, Ann. of Math. (2) **67** (1958), 517–573.
[P] A. V. Pogorelov, *Extrinsic geometry of convex surfaces*, Translations of Mathematical Monographs, Vol. 35., American Mathematical Society, Providence, R.I., 1973.
[RV] A. W. Roberts, D. E. Varberg, *Convex functions,* Pure and Applied Mathematics, 57, Academic Press, New York-London, 1973.
[S] J. J. Stoker, *Differential geometry*, Pure and Applied Mathematics, Vol. XX, Interscience Publishers, John Wiley & Sons, New York-London-Sydney, 1969.
[T] G. P. Tolstov, *Curves allowing a differentiable parametric representation* (Russian), Uspekhi Mat. Nauk **6** (1951), (43), 135–152.
[V] L. Veselý, *On the multiplicity points of monotone operators on separable Banach spaces*, Comment. Math. Univ. Carolin. **27** (1986), 551–570.
[VZ] L. Veselý, L. Zajíček, *Delta-convex mappings between Banach spaces and applications,* Dissertationes Math. (Rozprawy Mat.) **289** (1989).
[Z] Z. Zahorski, *On Jordan curves possessing a tangent at every point* (Russian), Mat. Sbornik (N.S.) **22 (64)** (1948), 3–26.

E-mail address: `jakub.duda@weizmann.ac.il`
E-mail address: `zajicek@karlin.mff.cuni.cz`

Department of Mathematics, Weizmann Institute of Science, Rehovot 76100, Israel

Charles University, Faculty of Mathematics and Physics, Sokolovská 83, 186 75 Praha 8-Karlín, Czech Republic

[1]Available electronically at `http://www.karlin.mff.cuni.cz/kma-preprints`.