Counterexample to prove that the Value-at-Risk is not subadditive

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Consider two assets, X and Y, that are usually normally distributed but are subject to occasional shocks. In particular, assume that X and Y are independent and identically distributed with

$$X = \varepsilon + \eta \text{ where } \varepsilon \sim N(0,1) \text{ and } \eta = \begin{cases} 0 & \text{w.p. } 0.991 \\ -10 & \text{w.p. } 0.009 \end{cases} \tag{1}$$

Consider a portfolio consisting of X and Y. Prove that

$$\text{V@R}_{0.99}(X+Y) = 9.8 > V@R_{0.99}(X) + V@R_{0.99}(Y) = 3.09 + 3.09 = 6.18 \tag{2}$$

Proof

Let's prove that $V@R_{0.99}(X) = 3.09$

$X \sim N(\mu,1) = \begin{cases} N(0,1) & \text{w.p. } 0.991 \\ N(-10,1) & \text{w.p. } 0.009 \end{cases}$

$P(X \leq q) = 0.01 =$

$= P(X \leq q|\mu = 0) \cdot P(\mu = 0) + P(X \leq q|\mu = -10) \cdot P(\mu = -10) =$

$= \Phi(q) \cdot 0.991 + \Phi(q+10) \cdot 0.009 = 0.01$

assume that $q = -5$, then $\Phi(q) \simeq 0, \Phi(q+10) \simeq 1$, then 0.009 too small, so it must be ≥ -5

assume that $q = 0$, then $\Phi(q) = 0.5, \Phi(q+10) \simeq 1$ then 0.5045 too large, so it must be ≤ 0

so q must be in a point between -5 and 0 where $\Phi(q+10) \simeq 1$ and $\Phi(q)$ is a specific value we need to find

$\Phi(q) \cdot 0.991 + 1 \cdot 0.009 = 0.01$

$\Phi(q) = \dfrac{0.001}{0.991}$

$q = -3.087546$

Let's prove that $V@R_{0.99}(X+Y) = 9.8$

$X+Y \sim N(\mu,2) \begin{cases} N(0,2) & \text{w.p. } 0.991^2 = 0.982 \\ N(-10,2) & \text{w.p. } 2 \cdot 0.991 \cdot 0.009 = 0.017838 \\ N(-20,2) & \text{w.p. } 0.009^2 = 0.000081 \end{cases}$

(here on Φ is inverse $N(0,2)$)

$P(X+Y \leq q) = 0.01 =$

$= P(X+Y \leq q|\mu = 0) \cdot P(\mu = 0) + P(X+Y \leq q|\mu = -10) \cdot P(\mu = -10) + P(X+Y \leq q|\mu = -20) \cdot P(\mu = -20) = 0.01$

$= \Phi(q) \cdot 0.982 + \Phi(q+10) \cdot 0.017838 + \Phi(q+20) \cdot 0.000081 = 0.01$

assume that $q = -15$, then $\Phi(q) \simeq 0, \Phi(q+10) \simeq 0, \Phi(q+20) \simeq 1$ then 0.00081 too small, so it must be ≥ -15

assume that $q = -5$, then $\Phi(q) \simeq 0, \Phi(q+10) \simeq 1, \Phi(q+20) \simeq 1$ then 0.017838 too large, so it must be ≤ -5

so q must be in a point between -15 and -5 where $\Phi(q+20) \simeq 1, \Phi(q) \simeq 0$ and $\Phi(q+10)$ is a specific value we need to find

$0 \cdot 0.982 + \Phi(q+10) \cdot 0.017838 + 1 \cdot 0.000081 = 0.01$

$\Phi(q+10) = \dfrac{0.00919}{0.017838}$

$q + 10 = 0.199$

$q = -9.801$