Numerical simulation of viscoelastic fluid described by Oldroyd-B model using finite element method and finite volume method

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1 Introduction

The simulation of viscoelastic fluids is a challenging problem. In this talk Oldroyd-B model is studied. The main problem appears at high Weissenberg number, where the simulation does not converge. The problem of high Weissenberg number was treated for example in [1] using log-conformation formulation. This formulation was used in [2] for the simulation of Oldroyd-B model using Finite element method. The same formulation was also successfully used in [3] using Finite volume method. In this talk another procedure is used, inspired by [4] the second order diminishing stabilization term is used both for Finite element and Finite Volume method, and both methods are compared.

2 Mathematical model

Oldroyd-B model is based on standard balance equations. We suppose that the fluid density ρ is constant. Then balance of mass and balance of linear momentum reduce to

$$\begin{aligned} \operatorname{div} \mathbf{v} &= 0 \\ \rho \dot{\mathbf{v}} &= \operatorname{div} \mathbf{T}, \end{aligned}$$

where $\mathbf{v}$ is fluid velocity, $\dot{\mathbf{v}}$ is a material time derivative of the velocity and $\mathbf{T}$ is a symmetric stress tensor (balance of angular momentum) in the form

$$\mathbf{T} = -p\mathbf{I} + 2\mu\mathbf{D} + G\mathbf{A}.$$

Pressure is denoted by p, μ is the dynamic viscosity, $\mathbf{D} = \left(\nabla\mathbf{v} + (\nabla\mathbf{v})^{\mathrm{T}}\right)/2$ is a symmetric part of the velocity gradient, G is the elastic modulus. The viscoelastic part of the stress tensor $\mathbf{A}$ satisfies the following partial differential equation

$$\mathbf{A} + \lambda \frac{\delta \mathbf{A}}{\delta t} = 2\lambda \mathbf{D},$$

where λ is the relaxation time. The derivative $\delta\mathbf{A}/\delta t$ stands for the objective time derivative and it can be chosen from the one-parametric family of Gordon-Schowalter derivatives

$$\left(\frac{\delta \mathbf{A}}{\delta t}\right) = \dot{\mathbf{A}} - \mathbf{W}\mathbf{A} + \mathbf{A}\mathbf{W} + a(\mathbf{D}\mathbf{A} + \mathbf{A}\mathbf{D}), \quad a \in [-1, 1],$$

where $\mathbf{W} = \left(\nabla\mathbf{v} - (\nabla\mathbf{v})^{\mathrm{T}}\right)/2$ is an antisymmetric part of the velocity gradient. For $a = -1$ this derivative is called upper convected derivative (used in Oldroyd-B model that is studied here), for $a = 1$ it is called lower convected derivative (Oldroyd-A model), and for $a = 0$ co-rotational derivative (co-rotational model)[1].

For Oldroyd-B model the viscoelastic part of the stress tensor satisfies

$$\mathbf{A} + \lambda \left(\frac{\partial \mathbf{A}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{A} - (\nabla \mathbf{v})\mathbf{A} - \mathbf{A}(\nabla \mathbf{v})^{\mathrm{T}} \right) = 2\lambda \mathbf{D}. \tag{1}$$

3 Computational domain

The computational domain is a two-dimensional pipe with several sinusoidal narrowings (see Fig. 1). Diameter of the pipe at the inlet is equal to D, diameter in the narrowest point is equal to D_{min}. It contains N_{seg} segments, the segment is a part of the pipe between two neighbouring narrowings, length of the segment is equal to L_{seg}. Part of the pipe that is straight at the inlet has the length L_{in}, a straight part at the outlet has the length L_{out}.

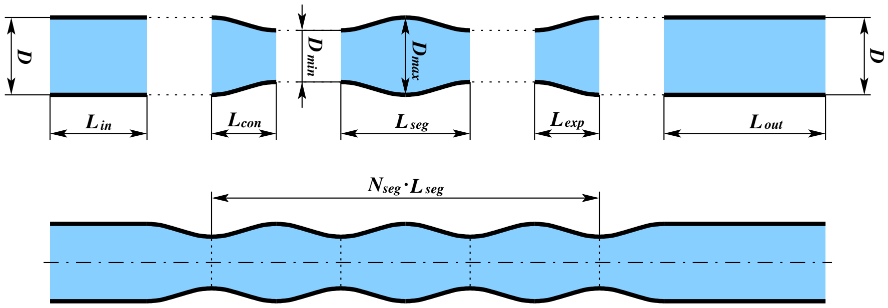


Figure 1: Computational domain.

The following material parameters are used: Density $\rho = 1000$ kg m^{-3}, dynamic viscosity $\mu = 9 \times 10^{-3}$ Pa s, elastic modulus $G = (10^{-3}/\lambda)$ Pa and the relaxation time λ controls the non-dimensional Weissenberg number

$$We = \frac{\lambda U}{D},$$

where U is the characteristic velocity and D the pipe diameter. The velocity at the inlet has a parabolic profile with the mean value $U = 10$ cm s^{-1}, pipe diameter at the inlet $D = 1$ cm. Stress-free boundary condition is prescribed at the outlet and no-slip boundary condition on the walls.

4 Numerical methods and results

Two different numerical methods are used – Finite element method and Finite volume method.

Finite element method is based on the weak solution of the governing equations. The implementation is based on the code developed in [5]. Pressure p/velocity $\mathbf{v}$/part of the stress $\mathbf{A}$ are approximated by $\mathrm{P1}^{\mathrm{disc}}$ /Q2/Q2 elements. A fully coupled monolithic finite element approach that treats all the numerical variables simultaneously is used. Set of linear algebraic equation is solved by the direct solver Umfpack.

[1]Note that there is no high Weissenberg number problem for the co-rotational model.

Finite volume method is implemented into OpenFOAM CFD toolbox. Central scheme second order upwind is used. Pressure implicit splitting operators algorithm is used, which means that first the linear momentum equation and it is solved and then is corrected by continuity equation in the special loop. The linear equations are solved using preconditioned Bi-Conjugated gradient with diagonal incomplete LU preconditioner.

For both methods further stabilization for the viscoelastic part of the stress tensor $\mathbf{A}$ is used. Instead of solving (1) equation

$$\mathbf{A} - k\Delta\mathbf{A} + \lambda\left(\frac{\partial\mathbf{A}}{\partial t} + \mathbf{v}\cdot\nabla\mathbf{A} - (\nabla\mathbf{v})\mathbf{A} - \mathbf{A}(\nabla\mathbf{v})^{\mathrm{T}}\right) = 2\lambda\mathbf{D}$$

is solved, where k is a stabilization coefficient.

When deriving viscoelastic Oldroyd-B model from microscopical principles, using statistical physics the laplace term emerges in Oldroyd-B model. Usually in the literature this term is omitted because the coefficient k is very small, according to [6] the coeficient $k \approx 10^{-9} - 10^{-7}$. Nevertheless the presence of the laplace term in the equations is important for the stabilization of the numerics.

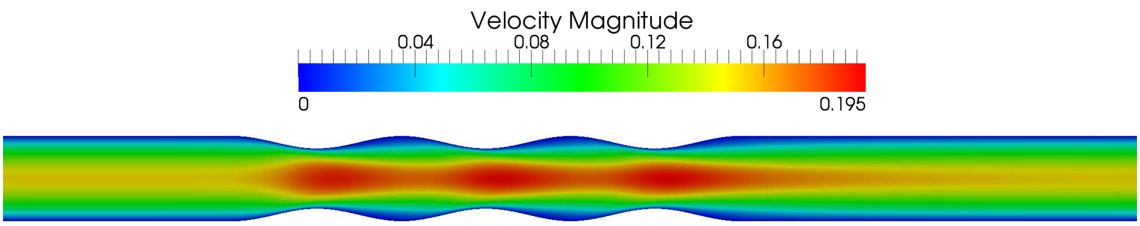
System of equations is solved in the following way: For $k = 1/2$ the result is obtained and it is used as an initial condition for a new computation with smaller k, this is repeated upto the k of order 10^{-8}. The process of decreasing the coefficient k can be done upto the critical value $k_{\min}$. For smaller k, the stabilization does not work and no solution can be found.

Using four different meshes with different mesh sizes h and six different Weissenberg numbers We it has been found out that

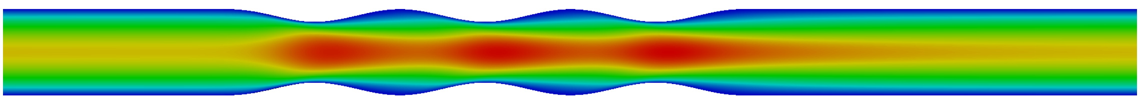
$$k_{\min} = Ch^2 \exp(We)$$

which means that the problem of high Weissenberg number can not be solved only by using denser meshes because the exponential function grows faster then the quadratic function.

The problem is simulated considering six different geometries – with 2, 3 and 4 segments and with two different lengths of segments – 2cm and 4cm. Some of the results are formulated in the conclusion. Comparison of Finite element and Finite Volume method for $N_{\mathrm{seg}} = 2, L_{\mathrm{seg}} = 2$ cm and $We = 0.5$ is depicted in Fig. 2.



Finite Element Method, $We = 0.5$



Finite Volume Method, $We = 0.5$

Figure 2: Comparison of FEM vs. FVM.

5 Conclusion

Both methods – Finite element and Finite volume method – show a good agreement which is an indicator that the computed results are correct. The results show that the problem highly depends on the geometry. Weissenberg number may not fully characterize the problem. With longer segments higher maximal Weissenberg number can be reached, on the other hand more segments cause a cumulation of the stress and so only lower maximal Weissenberg number is reached.

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