Infinite dimensional vector bundles and the module theory they inspire

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Drinfeld's question

"What is the appropriate definition of an (infinite–dimensional) vector bundle on a scheme X?"

V. Drinfeld,
Infinite–dimensional vector bundles in algebraic geometry: an introduction, The Unity of Mathematics, Birkhäuser, Boston 2006, 263–304.

Motto of Drinfeld's paper

"According to the Lamb conjecture, the key to the future development of Quantum Field Theory is probably buried in some forgotten paper published in the 30s. Attempts to follow up this conjecture, however, will probably be unsuccessful because of the Peierls-Jensen paradox; namely, that even if one finds the right paper, the point will probably be missed until it is found independently and accidentally by experiment."

"The Future of Field Theory," by Pure Imaginary Observer [PI]

From the Introduction

1.1 Subject of the article

The goal of this work is to show that there is a reasonable algebro-geometric notion of vector bundle with infinite-dimensional locally linearly compact fibers and that these objects appear “in nature.” Our approach is based on some results and ideas discovered in algebra during the period 1958-1972 by H. Bass, L. Gruson, I. Kaplansky, M. Karoubi, and M. Raynaud.

This article contains definitions and formulations of the main theorems, but practically no proofs. A detailed exposition will appear in [Dr].

. . .

The classes of modules involved I

Module $=$ right R–module (over an associative ring R with 1).

$\mathrm{Mod}\text{-}R$ $=$ the category of all modules.

$\mathcal{P}$ the class of all projective modules
$=$ direct summands of free modules
$=$ the modules P such that the functor $\mathrm{Hom}_R(P, -)$ is exact.

$\mathcal{F}$ the class of all flat modules
$=$ direct limits of free modules
$=$ the modules F such that the functor $F \otimes_R -$ is exact.

The classes of modules involved II

$\mathcal{L}$ the class of all Mittag–Leffler modules, i.e., the modules M such that the canonical map

$$M \otimes_R \prod_{i \in I} M_i \to \prod_{i \in I} (M \otimes_R M_i)$$

$$m \otimes_R (m_i)_{i \in I} \mapsto (m \otimes_R m_i)_{i \in I}$$

is monic for each family of left R–modules $(M_i \mid i \in I)$.

$\mathcal{D} = \mathcal{F} \cap \mathcal{L}$ the class of all flat Mittag-Leffler modules ("projective modules with a human face").

The affine scheme case

Let $X = \mathrm{Spec}(R)$ be an affine scheme (for a commutative ring R).

Theorem [Grothendieck–Serre]

The category $\mathfrak{Qco}(X)$ of all quasi–coherent sheaves on X is equivalent to $\mathrm{Mod}\text{-}R$.

In this equivalence, vector bundles on X correspond to finitely generated projective modules.

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Drinfeld's question reiterated

"What is the appropriate definition of an (infinite dimensional) vector bundle on a (non–affine) scheme X?"

An answer

[Drinfeld'2006]

Let X be a scheme.

A quasi–coherent sheaf $\mathfrak{F}$ is a vector bundle on X if for each open affine subset $\mathrm{Spec}(R) \subseteq X$, the R–module of sections $\Gamma(\mathrm{Spec(R)}, \mathfrak{F})$ is projective, but not necessarily finitely generated.

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E. E. Enochs, S. Estrada,
Relative homological algebra in the category of quasi–coherent sheaves,
Adv. in Math. 194(2005), 284–295.

E. E. Enochs, S. Estrada, J. R. García-Rozas,
Locally projective monoidal model structure for complexes of quasi–coherent sheaves on $\mathbf{P}^1(k)$, J. Lond. Math. Soc. 77(2008), 253–269.

An alternative answer

A 'slightly different notion' [Drinfeld'2006]

Replace "projective" by "flat Mittag–Leffler":

$\mathfrak{F}$ is a Drinfeld vector bundle if for each open affine subset $\mathrm{Spec}(R) \subseteq X$, $\Gamma(\mathrm{Spec(R)}, \mathfrak{F})$ is a flat Mittag–Leffler R–module.

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S. Estrada, P. Guil Asensio, M. Prest, J. T.,
Model category structures arising from Drinfeld vector bundles.
to appear in Advances in Math.

D. Herbera, J. T.,
Almost free modules and Mittag–Leffler conditions.
arXiv:0910.4277

A much more general notion

Replace “projective” by “flat”:

$\mathfrak{F}$ is a flat quasi–coherent sheaf if for each open affine set $\mathrm{Spec}(R) \subseteq X$, $\Gamma(\mathrm{Spec(R)}, \mathfrak{F})$ is a flat R–module.

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L. Alonso Tarrío, A. Jeremías López, J. Lipman,
Local homology and cohomology on schemes,
Ann. Sci. École Norm. Sup. 30(1997), 1–39.

D. Murfet, *Derived categories of quasi-coherent sheaves*,
PhD. Thesis.

J. Gillespie, *Kaplansky classes and derived categories*,
Math. Zeitschrift 257(2007), 811–843.

Structure of the modules involved: $\mathcal{P}$ and $\mathcal{F}$

$\mathcal{P} \subseteq \mathcal{D} \subseteq \mathcal{F} = \varinjlim \mathcal{P}$.

If any two of these classes coincide, then all coincide.
This happens, if and only if R is a right perfect ring
(i.e., R has dcc on principal left ideals).

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Theorem [Kaplansky'1958]

Each module in $\mathcal{P}$ is a direct sum of countably generated modules.

The class $\mathcal{F}$ is considered non–classifiable when R is not right perfect
(cf. [Melles'93] in the particular case of $R = \mathbb{Z}$).

The sandwich class $\mathcal{D}$

Théorème [Raynaud–Gruson'1971]

Let R be a ring and M a module. Then the following are equivalent:

- *M is a flat Mittag–Leffler module (i.e., $M \in \mathcal{D}$).*
- *Every finite (or countable) subset of M is contained in a countably generated projective submodule which is pure in M.*

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Example 1 — a hereditary case

An abelian group is $\aleph_1$–free if each of its countable subgroups is free.

[Azumaya–Facchini'1989] $\mathcal{D}$ = the class of all $\aleph_1$–free abelian groups in case $R = \mathbb{Z}$. Théorème follows by Pontryagin's Criterion from 1934.

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Example 2 — a non–hereditary case

[Herbera–T.] $\mathcal{D} =$ the class of all non–singular modules, in case R is the endomorphism ring of an infinite dimensional right linear space.

A test lemma

[Herbera–T.]

Let R be a ring, and M be a module such that $M = \varinjlim_{i \in I} M_i$ for a direct system of Mittag–Leffler modules $(M_i, f_{ij} \mid i < j \in I)$.

Assume that $\varinjlim_{n<\omega} M_{i_n}$ is Mittag–Leffler for each countable chain $i_0 < \cdots < i_n < i_{n+1} < \dots$ in I.

Then M is Mittag–Leffler.

Almost free modules and tensor products

Definition [Shelah'1981, Eklof–Mekler'2002]

Let R be a ring. A module M is $\aleph_1$–projective provided that there is a set $\mathcal{S}$ consisting of submodules of M such that

- each element of $\mathcal{S}$ is a countably generated projective module;
- each countable subset of M is contained in an element of $\mathcal{S}$;
- $\mathcal{S}$ is closed under unions of countable chains.

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Theorem [Herbera–T.]

Let R be a ring and M be a module.
Then M is flat Mittag–Leffler, if and only if M is $\aleph_1$–projective.

Transfinite extensions

Definition

Let R be a ring, M a module, and $\mathcal{C}$ a class of modules.
M is a transfinite extension of modules in $\mathcal{C}$ (or a $\mathcal{C}$–filtered module) if there is an increasing chain $(M_\alpha \mid \alpha \leq \sigma)$ of submodules of M such that

- $M_0 = 0$,
- $M_\alpha = \bigcup_{\beta<\alpha} M_\beta$ for each limit ordinal $\alpha \leq \sigma$,
- $M = M_\sigma$,
- $M_{\alpha+1}/M_\alpha$ is isomorphic to an element of $\mathcal{C}$ for each $\alpha < \sigma$.

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Examples

("Krull–Schmidt case") Direct sums of modules from $\mathcal{C}$ are $\mathcal{C}$–filtered.

("Jordan–Hölder case") If $\mathcal{C} = \text{simp-}R$, then $\mathcal{C}$–filtered $=$ semiartinian.

Closure properties

For a class of modules $\mathcal{C}$, let

$\mathcal{C}^\perp = \text{Ker Ext}^1_R(\mathcal{C}, -)$ and $^\perp\mathcal{C} = \text{Ker Ext}^1_R(-, \mathcal{C})$.

Eklof Lemma

$^\perp\mathcal{C}$ is closed under transfinite extensions, for each class of modules $\mathcal{C}$.

The classes $\mathcal{P}$, $\mathcal{D}$, and $\mathcal{F}$ are resolving classes closed under direct summands and transfinite extensions.

Moreover, $\mathcal{D}$ and $\mathcal{F}$ are closed under pure submodules.

Expanding a single transfinite extension into a large family

Hill Lemma

Let R be a ring, κ a regular infinite cardinal, and $\mathcal{E}$ a class of $< \kappa$–presented modules.
Let M be an $\mathcal{E}$–filtered module, with an $\mathcal{E}$–filtration $\mathcal{M} = (M_\alpha \mid \alpha \leq \sigma)$.

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Then there is a family $\mathcal{H}$ of submodules of M such that

- $\mathcal{M} \subseteq \mathcal{H}$.
- *$\mathcal{H}$ is closed under arbitrary sums and intersections; in fact, $\mathcal{H}$ is a distributive sublattice of the modular lattice of all submodules of M.*

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- *$\mathcal{H}$ is closed under arbitrary sums and intersections; in fact, $\mathcal{H}$ is a distributive sublattice of the modular lattice of all submodules of M.*
- *P/N is $\mathcal{E}$–filtered whenever $N \subseteq P$ are in $\mathcal{H}$.*
- *If $N \in \mathcal{H}$ and S is a subset of M of cardinality $< \kappa$, then there is $P \in \mathcal{H}$ with $N \cup S \subseteq P$ and P/N is $< \kappa$–presented.*

Quasi–coherent sheaves as qc–systems of modules

Theorem [Enochs–Estrada'2005]

Consider the quiver (V, E), where V is a covering family of open affine sets of X, and for $u, v \in V$, there is an edge $u \longrightarrow v$ in E, iff $v \subseteq u$. Then there is an equivalence between the category $\mathfrak{Qco}(X)$ and the category of certain representations of the quiver (V, E).

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Quasi–coherent sheaves on X $\iff$ qc–systems of modules.
The latter consist of sequences of $R(v)$–modules $\mathcal{M} = (M(v) \mid v \in V)$, where the $R(v)$s are the rings from the structure sheaf $\mathcal{O}_X$, and of systems of $R(u)$–homomorphisms $\rho_{uv} : M(u) \to M(v)$ for $u \to v$ in E, satisfying

- $id_{R(v)} \otimes \rho_{uv} : R(v) \otimes_{R(u)} M(u) \to R(v) \otimes_{R(u)} M(v) \cong M(v)$ *is an $R(v)$–isomorphism for each arrow $u \to v$ in E, and*
- *if $w \subseteq v \subseteq u$ in V, then $\rho_{uw} = \rho_{vw} \circ \rho_{uv}$.*

Corollary: $\mathfrak{Qco}(X)$ *is a Grothendieck category.*

The basic idea of [Quillen'1967] applied to $\mathfrak{Qco}(X)$

Rather than working in the derived category $\mathcal{D}(\mathfrak{Qco}(X))$ directly,

consider model structures on the category $\mathcal{C} = \mathsf{Ch}(\mathfrak{Qco}(X))$

of all unbounded complexes of quasi–coherent sheaves on the scheme X.

Model structures

Definition

A model structure on a complete and cocomplete category $\mathcal{C}$ consists of three distinguished classes of morphisms called *fibrations*, *cofibrations* and *weak equivalences* satisfying the following axioms:

1. If two of f, g, fg are weak equivalences, the so is the third.
2. Each of the three classes is closed under retracts.
3. *Lifting Properties*: Each trivial cofibration (resp. cofibration) has the left lifting property with respect to fibrations (resp. trivial fibrations).
4. *Functorial factorizations*: There are endofunctors $\alpha, \beta, \gamma, \delta$ of Map($\mathcal{C}$) such that for each morphism f of $\mathcal{C}$

$$f = \beta(f)\alpha(f) \begin{cases} \beta(f) \text{ triv.fibration} \\ \alpha(f) \text{ cofibration} \end{cases} \qquad f = \delta(f)\gamma(f) \begin{cases} \delta(f) \text{ fibration} \\ \gamma(f) \text{ triv.cof.} \end{cases}$$

Computing homology using a model structure

Theorem

Assume that all weak equivalences are homology isomorphisms. Then for all quasi–coherent sheaves M and N,

$$Ext^n_{\mathfrak{Qco}(X)}(M,N) = \mathrm{Hom}_{\mathcal{D}(\mathfrak{Qco}(X))}(S^0(M), S^n(N)) =$$

$$\mathrm{Hom}_{\mathbb{K}(\mathrm{Ch}(\mathfrak{Qco}(X)))}(Q_{S^0(M)}, P_{S^n(N)})$$

where

- $\mathbb{K}(\mathrm{Ch}(\mathfrak{Qco}(X)))$ *is the homotopy category associated to the model structure on* $\mathrm{Ch}(\mathfrak{Qco}(X))$.
- $S^n(M)$ *is the complex with M in the n-th position and* 0 *elsewhere.*
- $Q_{S^0(M)}$ *is the* cofibrant replacement *of* $S^0(M)$, *that is,* $0 \stackrel{\alpha(f)}{\rightarrow} Q_{S^0(M)} \stackrel{\beta(f)}{\rightarrow} S^0(M)$ *is the functorial filtration of* $0 \stackrel{f}{\rightarrow} S^0(M)$.
- $P_{S^n(N)}$ *is the* fibrant replacement *of* $S^n(N)$.

An example

Projective model structure in the affine scheme case

Let $X = \mathrm{Spec}(R)$ be an affine scheme (so $\mathfrak{Qco}(X) \simeq \mathrm{Mod}\text{-}R$).

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In the projective model structure on $\text{Ch}(\mathrm{Mod}\text{-}R)$,

- weak equivalences = homology isomorphisms,
- fibrations = surjections, and
- cofibrations = componentwise split injections with dg–projective cokernels.

Here, a complex X is *dg–projective* if X_n is projective for each n, and $\text{Hom}(X, E)$ is exact for all exact complexes E.

In this case, $\mathbb{K}(\text{Ch}(\mathfrak{Qco}(X))) = \mathcal{D}(\mathrm{Mod}\text{-}R)$.

Hovey's approach to model structures

The basic idea [Hovey'2002]

Compatible model structures on $\mathcal{C}$ correspond 1–1 to functorially complete cotorsion pairs on $\mathcal{C}$. The latter are easier to describe.

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Definition

A model structure on $\mathcal{C}$ is compatible if

- all cofibrations are monomorphisms, and
- a map is a (trivial) fibration iff it is an epimorphism with a (trivially) fibrant kernel.

Cotorsion pairs

Definition [Salce'1979]

A pair $(\mathcal{X}, \mathcal{Y})$ of classes of objects of $\mathcal{C}$ is a cotorsion pair provided that $\mathcal{X}^{\perp} = \mathcal{Y}$ and ${}^{\perp}\mathcal{Y} = \mathcal{X}$, where

$$\mathcal{X}^{\perp} = \{Y \in \mathcal{C} \mid \mathrm{Ext}^1_{\mathcal{C}}(X, Y) = 0 \quad \forall X \in \mathcal{X}\}.$$

$${}^{\perp}Y = \{X \in \mathcal{C} \mid \mathrm{Ext}^1_{\mathcal{C}}(X, Y) = 0 \quad \forall Y \in \mathcal{Y}\}.$$

A cotorsion pair $(\mathcal{X}, \mathcal{Y})$ is hereditary if $\mathrm{Ext}^i_{\mathcal{C}}(X, Y) = 0$ for all $i \geq 2$.

A cotorsion pair $(\mathcal{X}, \mathcal{Y})$ is small (or *generated by a set*) if there is a set of objects $\mathcal{S} \subseteq \mathcal{C}$ containing a generator of $\mathcal{C}$ such that $\mathcal{Y} = \mathcal{S}^{\perp}$.

Functorially complete cotorsion pairs

A cotorsion pair $(\mathcal{X}, \mathcal{Y})$ is functorially complete if for each $C \in \mathcal{C}$ there exists short exact sequences

$$0 \longrightarrow Y \longrightarrow X \longrightarrow C \longrightarrow 0$$

and

$$0 \longrightarrow C \longrightarrow Y' \longrightarrow X' \longrightarrow 0$$

with $X, X' \in \mathcal{X}$ and $Y, Y' \in \mathcal{Y}$,
and the assignments $C \mapsto X$ and $C \mapsto Y'$ can be taken functorial.

Theorem

Each small cotorsion pair in $\mathcal{C}$ is functorially complete.

The correspondence

Theorem [Hovey 2002]

Let $\mathcal{F}$, $\mathcal{G}$ and $\mathcal{T}$ be classes in $\mathcal{C}$ such that

1. $\mathcal{T}$ *is* thick, *that is,* $\mathcal{T}$ *is closed under retracts, and if two terms of a short exact sequence in* $\mathcal{C}$ *belong to* $\mathcal{T}$*, so does the third.*
2. $(\mathcal{G}, \mathcal{F} \cap \mathcal{T})$ *and* $(\mathcal{G} \cap \mathcal{T}, \mathcal{F})$ *are functorially complete (eg., small) cotorsion pairs in* $\mathcal{C}$*.*

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Then there is a unique compatible model structure on $\mathcal{C}$ *such that* $\mathcal{F}$ *is the class of all fibrant objects,* $\mathcal{G}$ *is the class of all cofibrant objects, and* $\mathcal{T}$ *is the class of all trivial objects.*

Conversely, given a compatible model structure on $\mathcal{C}$, its fibrant, cofibrant and trivial objects satisfy the conditions (1) and (2) above.

The projective model structure in the affine scheme case II

$X = \mathrm{Spec}(R)$ and $\mathfrak{Qco}(X) \simeq \mathrm{Mod}\text{-}R$.

In the projective model structure on $\mathrm{Ch}(\mathrm{Mod}\text{-}R)$,

- $\mathcal{T}$ = all exact complexes,
- $\mathcal{F}$ = all complexes, and
- $\mathcal{G}$ = all dg–projective complexes.

Some terminology for complexes

- Given a complex (X, d^X)

$$\cdots \xrightarrow{d^X_{n+2}} X_{n+1} \xrightarrow{d^X_{n+1}} X_n \xrightarrow{d^X_n} X_{n-1} \xrightarrow{d^X_{n-1}} \cdots$$

  we shall denote
  - $Z_nX = \mathrm{Ker}\ d_n$ (the nth–syzygy).
- **The *Hom*–complex**: $H = \mathrm{Hom}(X, Y)$
  - $H_n = \prod_{k \in \mathbb{Z}} \mathrm{Hom}_{\mathcal{C}}(X_k, Y_{k+n})$
  - $\left((f)d^H_n\right)_k = f_k d^Y_{k+n} - (-1)^n d^X_k f_{k-1}$.
- **All exact complexes**: $\mathcal{E}$.

Applications of Hovey's correspondence

- Let $\mathcal{C}'$ be the category of all unbounded complexes of modules over a ring R.
- Let $(\mathcal{A}, \mathcal{B})$ be a small hereditary cotorsion pair in $\text{Mod-}R$ or $\mathfrak{Qco}(X)$.
- Define the following classes of complexes (in $\mathcal{C}$ or $\mathcal{C}'$):
  1. $\mathcal{A}$-complexes: $\tilde{\mathcal{A}} = \{X \in \mathcal{E} \mid Z_n X \in \mathcal{A} \text{ for all } n\}$.
  2. $\mathcal{B}$-complexes: $\tilde{\mathcal{B}} = \{X \in \mathcal{E} \mid Z_n X \in \mathcal{B} \text{ for all } n\}$.
  3. dg–$\mathcal{A}$ complexes:
     $\text{dg}\tilde{\mathcal{A}} = \{X \mid X_n \in \mathcal{A} \text{ for all } n \text{ and } \text{Hom}(X, Y) \in \mathcal{E} \text{ for all } Y \in \tilde{\mathcal{B}}\}$.
  4. dg–$\mathcal{B}$ complexes:
     $\text{dg}\tilde{\mathcal{B}} = \{X \mid X_n \in \mathcal{B} \text{ for all } n \text{ and } \text{Hom}(Y, X) \in \mathcal{E} \text{ for all } Y \in \tilde{\mathcal{A}}\}$.

The idea is to find conditions under which the classes $\mathcal{E}$, $dg\tilde{\mathcal{A}}$ and $dg\tilde{\mathcal{B}}$ satisfy the hypotheses of Hovey's Theorem.

A module–theoretic warm up

Theorem [Gillespie 2004]

Let $(\mathcal{A}, \mathcal{B})$ be a small hereditary cotorsion pair in Mod-R.
If $\mathcal{A}$ is closed under direct limits, *then there exists a compatible model structure on $\mathcal{C}'$ such that:*

- *the weak equivalences are the homology isomorphisms;*
- *the cofibrations (resp. trivial cofibrations) are the monomorphisms whose cokernels are in $dg\tilde{\mathcal{A}}$ (resp. $\tilde{\mathcal{A}}$), and*
- *the fibrations (resp. trivial fibrations) are the epimorphisms whose kernels are in $dg\tilde{\mathcal{B}}$ (resp. $\tilde{\mathcal{B}}$).*

Case I : Flat quasi–coherent sheaves

Theorem [Gillespie'2007]

Assume X is a scheme with a generating set of flat quasi–coherent sheaves (e.g., X is quasi–compact and semi–separated).
Then there is a compatible model structure on $\mathcal{C}$ where weak equivalences are homology isomorphisms, and cofibrations are monomorphisms whose cokernels are dg–flat complexes of quasi–coherent sheaves.

Proof: Makes use of Hovey's correspondence;
Essential point: $\mathcal{F}$ is closed under direct limits.

Case II : Vector bundles

Theorem [Enochs–Estrada–García-Rozas'2008]

If X is the projective line then there is a compatible model structure on $\mathcal{C}$ such that weak equivalences are homology isomorphisms, and cofibrations are monomorphisms whose cokernels are dg–complexes of vector bundles.

Proof: Based on Grothendieck's theorem on the structure of finite dimensional vector bundles over the projective line.

The general case

Theorem [Estrada–Guil–Prest–T.]

Let X be a semi–separated scheme, and κ be an infinite cardinal such that $\kappa \geq \operatorname{card} V$, and $R(v)$ is κ–noetherian for all $v \in V$. For each $v \in V$, let S_v be any *syzygy closed set of $\leq \kappa$–presented $R(v)$–modules. Let $\mathcal{A}$ denote the class of all quasi-coherent sheaves such that $M(v) \in {}^{\perp}(S_v^{\perp})$ for all $v \in V$. Assume also that $\mathcal{A}$ contains a set of generators of $\mathcal{C}$.*

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Then there is a compatible model structure on $\mathcal{C}$ where weak equivalences are homology isomorphisms, and cofibrations are monomorphisms whose cokernels are $dg\tilde{A}$–complexes of quasi–coherent sheaves.

The general case

Theorem [Estrada–Guil–Prest–T.]

Let X be a semi–separated scheme, and κ be an infinite cardinal such that $\kappa \geq \operatorname{card} V$, and $R(v)$ is κ–noetherian for all $v \in V$. For each $v \in V$, let S_v be any *syzygy closed set of $\leq \kappa$–presented $R(v)$–modules. Let $\mathcal{A}$ denote the class of all quasi-coherent sheaves such that $M(v) \in {}^{\perp}(S_v^{\perp})$ for all $v \in V$. Assume also that $\mathcal{A}$ contains a set of generators of $\mathcal{C}$.*

Then there is a compatible model structure on $\mathcal{C}$ where weak equivalences are homology isomorphisms, and cofibrations are monomorphisms whose cokernels are $dg\tilde{A}$–complexes of quasi–coherent sheaves.

Proof: Makes use of Hovey's correspondence; Key idea:
apply the Hill Lemma as a tool for making local filtrations compatible.

Corollaries

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Flat quasi–coherent sheaves

Gillespie's theorem follows for S_v = a representative set of all $\leq \kappa$–generated flat $R(v)$–modules ($v \in V$, κ sufficiently large).

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Vector bundles on semi–separated schemes

If $\mathcal{C}$ has a generating set of vector bundles, then there is a compatible model structure on $\mathcal{C}$ such that weak equivalences are homology isomorphisms, and cofibrations are monomorphisms whose cokernels are dg–complexes of vector bundles.

Proof: Set $S_v = \{R(v), 0\}$ for each $v \in V$.

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Proof: Set $S_v = \{R(v), 0\}$ for each $v \in V$.

A generalization: κ–restricted Drinfeld vector bundles

For each large enough infinite cardinal κ, we take S_v = a representative set of all $\leq \kappa$–generated $\aleph_1$–projective $R(v)$–modules.

What about the "unrestricted" Drinfeld vector bundles?

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Definition [Eklof'2006]

A class of modules $\mathcal{A}$ is deconstructible in case there is a cardinal κ such that each $M \in \mathcal{A}$ is a transfinite extension of $< \kappa$–presented modules in $\mathcal{A}$.

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The crucial question: Is there a subset $\mathcal{S} \subseteq \mathcal{D}$ such that $\mathcal{D} = {}^{\perp}(\mathcal{S}^{\perp})$?
Equivalently: Is there a small cotorsion pair of the form $(\mathcal{D}, \mathcal{D}^{\perp})$?
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Still equivalently: **Is $\mathcal{D}$ deconstructible?**

Theorem [Herbera–T.]

The class $\mathcal{D}$ is deconstructible, if and only if R is a right perfect ring.

So Hovey's approach applies to all vector bundles, flat quasi–coherent sheaves, and "restricted" Drinfeld vector bundles, but **not** to "unrestricted" Drinfeld vector bundles.

Definition

Let R be a ring. A class of modules $\mathcal{A}$ is precovering if for each module M there is $f \in \mathrm{Hom}_R(A, M)$ with $A \in \mathcal{C}$ such that each $f' \in \mathrm{Hom}_R(A', M)$ with $A' \in \mathcal{A}$ has a factorization through f:

$$\begin{array}{ccc} A & \xrightarrow{f} & M \\ \uparrow & \nearrow_{f'} & \\ A' & & \end{array} .$$

Approximations of modules

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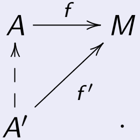

If $\mathcal{A}$ is deconstructible and closed under transfinite extensions, then $\mathcal{A}$ is precovering.

In particular, the classes $\mathcal{P}$ and $\mathcal{F}$ are precovering.

A failure of functorial completeness for $\mathfrak{Qco}(\mathrm{Spec}(R))$

Theorem [Eklof–Shelah'2003], [Estrada–Guil–Prest–T.], [Šaroch–T.], [Bazzoni–Šťovíček]

Let R be a countable non–right perfect ring.

Then the class $\mathcal{D}$ of all $\aleph_1$–projective modules is **not** *precovering.*

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Let R be a countable non–right perfect ring.

Then the class $\mathcal{D}$ of all $\aleph_1$–projective modules is **not** *precovering.*

There is no functorially complete cotorsion pair of the form $(\mathcal{D}, \mathcal{D}^{\perp})$ in the category $\mathcal{C} = \mathfrak{Qco}(X)$ for the affine scheme $X = \mathrm{Spec}(R)$ where R is any countable commutative noetherian ring of Krull dimension ≥ 1.

So in general, there is **no** compatible model structure induced by the class of **all** Drinfeld vector bundles.

Open problems

Two conjectures concerning flat Mittag–Leffler modules

1. ${}^{\perp}(\mathcal{D}^{\perp}) = \mathcal{F}$ holds for any ring R.

2. The class $\mathcal{D}$ is precovering, if and only if R is a right perfect ring.

Presently, these Conjectures are proved only for countable rings ...