Approximations and Locally Free Modules

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[Gruson-Jensen'73], [Huisgen-Zimmermann'79]
Mod-R is decomposable, iff R is right pure-semisimple.
Uniformly: $\kappa = \aleph_0$ sufficient for all such R;
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Let $\mathcal{C} \subseteq$ Mod-R. A module M is $\mathcal{C}$-filtered (or a transfinite extension of the modules in $\mathcal{C}$), provided that there exists an increasing sequence $(M_\alpha \mid \alpha \leq \sigma)$ consisting of submodules of M such that $M_0 = 0$, $M_\sigma = M$,

- $M_\alpha = \bigcup_{\beta<\alpha} M_\beta$ for each limit ordinal $\alpha \leq \sigma$, and
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Eklof Lemma

The class ${}^\perp\mathcal{C} := \text{KerExt}^1_R(-, \mathcal{C})$ is closed under transfinite extensions for each class of modules $\mathcal{C}$.

In particular, so are the classes $\mathcal{P}_n$ and $\mathcal{F}_n$ of all modules of projective and flat dimension $\leq n$, for each $n < \omega$.

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A class of modules $\mathcal{A}$ is deconstructible, provided there is a cardinal κ such that $\mathcal{A} \subseteq \text{Filt}(\mathcal{A}^{<\kappa})$, where $\mathcal{A}^{<\kappa}$ denotes the class of all $< \kappa$-presented modules from $\mathcal{A}$.

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[Eklof-T.'01], [Šťovíček-T.'09]

For each set of modules $\mathcal{S}$, the class ${}^{\perp}(\mathcal{S}^{\perp})$ is deconstructible.
Here, $\mathcal{S}^{\perp} := \mathrm{KerExt}^1_R(\mathcal{S}, -)$.

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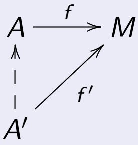
A class of modules $\mathcal{A}$ is precovering if for each module M there is $f \in \mathrm{Hom}_R(A, M)$ with $A \in \mathcal{A}$ such that each $f' \in \mathrm{Hom}_R(A', M)$ with $A' \in \mathcal{A}$ has a factorization through f:



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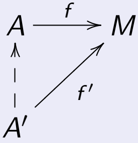

The map f is called an $\mathcal{A}$–precover of M.

[Saorín-Šťovíček'11], [Enochs'12]

All deconstructible classes closed under transfinite extensions are precovering.

In particular, so are the classes ${}^{\perp}(\mathcal{S}^{\perp})$ for all sets of modules $\mathcal{S}$.

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- [Eklof-Shelah'03] It is independent of ZFC whether the class of all Whitehead groups ($= {}^{\perp}\{\mathbb{Z}\}$) is precovering (or deconstructible).

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A result in ZFC

A module M is flat Mittag-Leffler provided the functor $M \otimes_R -$ is exact, and for each system of left R-modules $(N_i \mid i \in I)$, the canonical map $M \otimes_R \prod_{i \in I} N_i \to \prod_{i \in I} M \otimes_R N_i$ is monic.

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Assume that R is not right perfect.

- [Herbera-T.'12] The class $\mathcal{FM}$ of all flat Mittag-Leffler modules is closed under transfinite extensions, but it is not deconstructible.
- [Šaroch-T.'12], [Bazzoni-Šťovíček'12] If R is countable, then $\mathcal{FM}$ is not precovering.

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Definition

A module M is locally $\mathcal{F}$-free, if M possesses a subset $\mathcal{S}$ consisting of countably $\mathcal{F}$-filtered modules, such that

- each countable subset of M is contained in an element of $\mathcal{S}$,
- $0 \in \mathcal{S}$, and $\mathcal{S}$ is closed under unions of countable chains.

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Note: If M is countably generated, then M is locally $\mathcal{F}$-free, iff M is countably $\mathcal{F}$-filtered.

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Lemma (Slávik-T.)

- *$\mathcal{L}$ is closed under transfinite extensions.*
- $\mathcal{L}^{\perp} \subseteq (\varinjlim_{\omega} \mathcal{F})^{\perp}$.

Basic example: $\mathcal{FM}$ as a particular instance of $\mathcal{L}$

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Theorem (Herbera-T.'12)

Let $\mathcal{F} =$ be the class of all countably presented projective modules. Then the notions of a locally $\mathcal{F}$-free module and a flat Mittag-Leffler module coincide for any ring R.

Basic example: $\mathcal{FM}$ as a particular instance of $\mathcal{L}$

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Example

Let $R = \mathbb{Z}$. An abelian group A is flat Mittag-Leffler, iff all countable subgroups of A are free.
In particular, the Baer-Specker group $\mathbb{Z}^\kappa$ is flat Mittag-Leffler for each κ, but not free.

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- $\mathcal{F}$ a class of countably presented modules,
- $\mathcal{L}$ the class of all locally $\mathcal{F}$-free modules,
- $\mathcal{D}$ the class of all direct summands of the modules M that fit into an exact sequence

$$0 \to F' \to M \to C' \to 0,$$

where F' is a free module, and C' is countably $\mathcal{F}$-filtered.

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In particular, the class $\mathcal{FM}$ is not deconstructible for each non-right perfect ring R.

Infinite dimensional tilting modules

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T is a tilting module provided that

- T has finite projective dimension,
- $\mathrm{Ext}^i_R(T, T^{(\kappa)}) = 0$ for each cardinal κ, and
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$\mathcal{A}_T := {}^{\perp}\mathcal{B}_T$ the left tilting class of T.

Theorem (A characterization of right tilting classes)

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The tilting module T is $\sum$-pure split provided that $\bar{\mathcal{A}}_T = \mathcal{A}_T$, that is, the left tilting class of T is closed under direct limits. Equivalently: Each pure embedding $T_0 \hookrightarrow T_1$ such that $T_0, T_1 \in \text{Add}(T)$ splits.

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Example

Let $T = R$. Then T is a tilting module of projective dimension 0, and T is $\sum$-pure split, iff R is a right perfect ring.

Locally free modules and tilting

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The setting

Let R be a countable ring, and T be a non-$\sum$-pure-split tilting module. Let $\mathcal{F}_T$ be the class of all countably presented modules in $\mathcal{A}_T$, and $\mathcal{L}_T$ the class of all locally $\mathcal{F}_T$-free modules.

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Assume that $\mathcal{L}_T \subseteq \mathcal{P}_1$, $\mathcal{L}_T$ is closed under direct summands, and $M \in \mathcal{L}_T$ whenever $M \subseteq L \in \mathcal{L}_T$ and $L/M \in \bar{\mathcal{A}}_T$.
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Corollary

If R is countable and non-right perfect, then $\mathcal{FM}$ is not precovering.

An different class of examples

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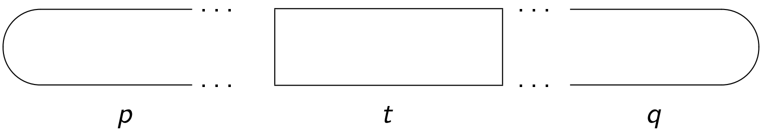
Let R be an indecomposable hereditary artin algebra of infinite representation type, with the Auslander-Reiten translation τ.
Then there is a partition of all indecomposable finitely generated modules into three sets:

q := indecomposable preinjective modules
(i.e., indecomposable injectives and their τ-shifts),

p := indecomposable preprojective modules
(i.e., indecomposable projectives and their τ^--shifts),

t := regular modules (the rest).



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[Angeleri-Kerner-T.'10]

The class of all Baer modules coincides with $\mathrm{Filt}(p)$.

The Lukas tilting module L is countably generated, but has no finite dimensional direct summands, and it is not $\sum$-pure split.

Non-deconstructibility in the realm of artin algebras

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Theorem (Slávik-T.)

Let R be a countable indecomposable hereditary artin algebra of infinite representation type. Then the class of all locally Baer modules is not precovering (and hence not deconstructible).

A conjecture

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A ring R is right pure-semisimple, iff each class of right R-modules closed under transfinite extensions and direct summands is deconstructible.