New tools of set-theoretic homological algebra and their applications to modules

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Classic structure theory: direct sum decompositions

A class of modules $\mathcal{C}$ is decomposable, provided that there is a cardinal κ such that each module in $\mathcal{C}$ is a direct sum of strongly $< \kappa$-presented modules from $\mathcal{C}$.

Examples

1. (Kaplansky) The class $\mathcal{P}_0$ of all projective modules is decomposable.

2. (Faith-Walker) The class $\mathcal{I}_0$ of all injective modules is decomposable iff R is a right noetherian ring.

3. (Huisgen-Zimmermann) Mod-R is decomposable iff R is a right pure-semisimple ring. In fact, if M is a module such that Prod(M) is decomposable, then M is Σ-pure-injective.

Note: Krull-Schmidt type theorems hold in the cases 2. and 3.

Such examples, however, are rare in general – most classes of modules are not decomposable.

Example

Assume that the ring R is not right perfect, that is, there is a strictly decreasing chain of principal left ideals

$$Ra_0 \supsetneq \cdots \supsetneq Ra_n \dots a_0 \supsetneq Ra_{n+1}a_n \dots a_o \supsetneq \dots$$

Then the class $\mathcal{F}_0$ of all flat modules is not decomposable.

Example

There exist arbitrarily large indecomposable flat abelian groups.

Transfinite extensions

Let $\mathcal{A} \subseteq \mathrm{Mod}\text{–}R$. A module M is $\mathcal{A}$-filtered (or a transfinite extension of the modules in $\mathcal{A}$), provided that there exists an increasing sequence $(M_\alpha \mid \alpha \leq \sigma)$ consisting of submodules of M such that $M_0 = 0$, $M_\sigma = M$,

- $M_\alpha = \bigcup_{\beta < \alpha} M_\beta$ for each limit ordinal $\alpha \leq \sigma$, and
- for each $\alpha < \sigma$, $M_{\alpha+1}/M_\alpha$ is isomorphic to an element of $\mathcal{A}$.

Notation: $M \in \mathrm{Filt}(\mathcal{A})$. A class $\mathcal{A}$ is filtration closed if $\mathrm{Filt}(\mathcal{A}) = \mathcal{A}$.

Eklof Lemma

${}^\perp\mathcal{C} = \mathrm{KerExt}^1_R(-, \mathcal{C})$ is filtration closed for each class of modules $\mathcal{C}$.

In particular, so are the classes $\mathcal{P}_n$ and $\mathcal{F}_n$ of all modules of projective and flat dimension $\leq n$, for each $n < \omega$.

Deconstructible classes

A class of modules $\mathcal{A}$ is deconstructible, provided there is a cardinal κ such that $\mathcal{A} = \mathrm{Filt}(\mathcal{A}^{<\kappa})$ where $\mathcal{A}^{<\kappa}$ denotes the class of all strongly $< \kappa$-presented modules from $\mathcal{A}$.

All decomposable classes closed under direct summands are deconstructible.

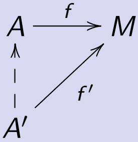
For each $n < \omega$, the classes $\mathcal{P}_n$ and $\mathcal{F}_n$ are deconstructible.

[Eklof-T.]

More in general, for each set of modules $\mathcal{S}$, the class ${}^{\perp}(\mathcal{S}^{\perp})$ is deconstructible. Here, $\mathcal{S}^{\perp} = \mathrm{KerExt}^1_R(\mathcal{S}, -)$.

Approximations for relative homological algebra

A class of modules $\mathcal{A}$ is precovering if for each module M there is $f \in \mathrm{Hom}_R(A, M)$ with $A \in \mathcal{A}$ such that each $f' \in \mathrm{Hom}_R(A', M)$ with $A' \in \mathcal{A}$ factorizes through f:



The map f is an $\mathcal{A}$–precover of M.

If f is moreover right minimal (that is, f factorizes through itself only by an automorphism of A), then f is an $\mathcal{A}$–cover of M.

If $\mathcal{A}$ provides for covers for all modules, then $\mathcal{A}$ is called a covering class.

The abundance of approximations

[Enochs]

Each precovering class closed under direct limits is covering.

[Enochs], [Šťovíček]

All deconstructible classes are precovering.

In particular, the class ${}^{\perp}(\mathcal{S}^{\perp})$ is precovering for any set of modules $\mathcal{S}$.
Note: If $R \in \mathcal{S}$, then ${}^{\perp}(\mathcal{S}^{\perp})$ coincides with the class of all direct summands of $\mathcal{S}$-filtered modules.

Flat cover conjecture

$\mathcal{F}_0$ is covering for any ring R, and so are the classes $\mathcal{F}_n$ for each $n > 0$.

The classes $\mathcal{P}_n$ ($n \geq 0$) are precovering. ...

Bass modules

Let R be a ring and $\mathcal{F}$ be a class of finitely presented modules.
$\varinjlim_\omega \mathcal{F}$ denotes the class of all Bass modules over $\mathcal{F}$, that is, the modules B that are countable direct limits of modules from $\mathcal{F}$.
W.l.o.g., such B is the direct limit of a chain

$$F_0 \overset{f_0}{\to} F_1 \overset{f_1}{\to} \dots \overset{f_{i-1}}{\to} F_i \overset{f_i}{\to} F_{i+1} \overset{f_{i+1}}{\to} \dots$$

with $F_i \in \mathcal{F}$ and $f_i \in \mathrm{Hom}_R(F_i, F_{i+1})$ for all $i < \omega$.

The classic Bass module

Let $\mathcal{F}$ be the class of all finitely generated projective modules. Then the Bass modules coincide with the countably presented flat modules.
If R is not right perfect, then a classic instance of such a Bass module B arises when $F_i = R$ and f_i is the left multiplication by a_i ($i < \omega$) where $Ra_0 \supsetneq \dots \supsetneq Ra_n \dots a_0 \supsetneq Ra_{n+1}a_n \dots a_o \supsetneq \dots$ is strictly decreasing.

Flat Mittag-Leffler modules

[Raynaud-Gruson]

A module M is flat Mittag-Leffler provided the functor $M \otimes_R -$ is exact, and for each system of left R-modules $(N_i \mid i \in I)$, the canonical map $M \otimes_R \prod_{i \in I} N_i \to \prod_{i \in I} M \otimes_R N_i$ is monic.

The class of all flat Mittag-Lefler modules is denoted by $\mathcal{FM}$.

$\mathcal{P}_0 \subseteq \mathcal{FM} \subseteq \mathcal{F}_0$.
$\mathcal{FM}$ is filtration closed and closed under pure submodules.

$M \in \mathcal{FM}$, iff each countable subset of M is contained in a countably generated projective and pure submodule of M.
In particular, all countably generated modules in $\mathcal{FM}$ are projective.

Flat Mittag-Leffler modules and approximations

Theorem

Assume that R is not right perfect. Then the class $\mathcal{FM}$ is not precovering, and hence not deconstructible.

Idea of proof: Choose a non-projective Bass module B over $\mathcal{P}_0^{<\omega}$, and prove that B has no $\mathcal{FM}$-precover.

The main tool: Tree modules.

The trees

Let κ be an infinite cardinal, and T_κ be the set of all finite sequences of ordinals $< \kappa$, so

$$T_\kappa = \{\tau : n \to \kappa \mid n < \omega\}.$$

Partially ordered by inclusion, T_κ is a tree, called the tree on κ.

Let $\mathrm{Br}(T_\kappa)$ denote the set of all branches of T_κ. Each $\nu \in \mathrm{Br}(T_\kappa)$ can be identified with an ω-sequence of ordinals $< \kappa$:

$$\mathrm{Br}(T_\kappa) = \{\nu : \omega \to \kappa\}.$$

$|T_\kappa| = \kappa$ and $|\mathrm{Br}(T_\kappa)| = \kappa^\omega$.

Notation: $\ell(\tau)$ denotes the length of τ for each $\tau \in T_\kappa$.

Decorating trees by Bass modules

Let $D := \bigoplus_{\tau \in T_\kappa} F_{\ell(\tau)}$, and $P := \prod_{\tau \in T_\kappa} F_{\ell(\tau)}$.

For $\nu \in \mathrm{Br}(T_\kappa)$, $i < \omega$, and $x \in F_i$, we define $x_{\nu i} \in P$ by

$$\pi_{\nu \restriction i}(x_{\nu i}) = x,$$

$$\pi_{\nu \restriction j}(x_{\nu i}) = g_{j-1} \dots g_i(x) \text{ for each } i < j < \omega,$$

$$\pi_\tau(x_{\nu i}) = 0 \text{ otherwise},$$

where $\pi_\tau \in \mathrm{Hom}_R(P, F_{\ell(\tau)})$ denotes the τth projection for each $\tau \in T_\kappa$.

Let $X_{\nu i} := \{x_{\nu i} \mid x \in F_i\}$. Then $X_{\nu i}$ is a submodule of P isomorphic to F_i.

The tree modules

Let $X_\nu := \sum_{i<\omega} X_{\nu i}$, and $G := \sum_{\nu \in \mathrm{Br}(T_\kappa)} X_\nu$.

Basic properties

- $D \subseteq G \subseteq P$.
- There is a 'tree module' exact sequence

$$0 \to D \to G \to B^{(\mathrm{Br}(T_\kappa))} \to 0.$$

- G is a flat Mittag-Leffler module.

Proof of the Theorem

Assume there exists a $\mathcal{FM}$-precover $f : F \to B$ of the classic Bass module B. Let $K = \mathrm{Ker}(f)$, so we have an exact sequence

$$0 \to K \hookrightarrow F \xrightarrow{f} B \to 0.$$

Let κ be an infinite cardinal such that $|R| \leq \kappa$ and $|K| \leq 2^\kappa = \kappa^\omega$.

Consider the 'tree module' exact sequence

$$0 \to D \hookrightarrow G \to B^{(2^\kappa)} \to 0,$$

so $G \in \mathcal{FM}$ and D is a free module of rank κ. Clearly, $G \in \mathcal{P}_1$.

Let $\eta : K \to E$ be a $\{G\}^\perp$-preenvelope of K with a $\{G\}$-filtered cokernel.

Consider the pushout

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & K & \xrightarrow{\subseteq} & F & \xrightarrow{f} & B & \longrightarrow & 0 \\
 & & \eta \downarrow & & \varepsilon \downarrow & & \| & & \\
0 & \longrightarrow & E & \xrightarrow{\subseteq} & P & \xrightarrow{g} & B & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & \mathrm{Coker}(\eta) & \xrightarrow{\cong} & \mathrm{Coker}(\varepsilon) & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}
$$

Then $P \in \mathcal{FM}$. Since f is an $\mathcal{FM}$-precover, there exists $h : P \to F$ such that $fh = g$. Then $f = g\varepsilon = fh\varepsilon$, whence $K + \mathrm{Im}(h) = F$. Let $h' = h \restriction E$. Then $h' : E \to K$ and $\mathrm{Im}(h') = K \cap \mathrm{Im}(h)$.

Consider the restricted exact sequence

$$0 \longrightarrow \mathrm{Im}(h') \xrightarrow{\subseteq} \mathrm{Im}(h) \xrightarrow{f\restriction \mathrm{Im}(h)} B \longrightarrow 0.$$

As $E \in G^{\perp}$ and $G \in \mathcal{P}_1$, also $\mathrm{Im}(h') \in G^{\perp}$.

Applying $\mathrm{Hom}_R(-, \mathrm{Im}(h'))$ to the 'tree-module' exact sequence above, we obtain the exact sequence

$$\mathrm{Hom}_R(D, \mathrm{Im}(h')) \to \mathrm{Ext}^1_R(B, \mathrm{Im}(h'))^{2^\kappa} \to 0$$

where the first term has cardinality $\leq |K|^\kappa \leq 2^\kappa$, so the second term must be zero.

This yields $\mathrm{Im}(h') \in B^{\perp}$. Then $f \restriction \mathrm{Im}(h)$ splits, and so does the $\mathcal{FM}$-precover f, a contradiction with $B \notin \mathcal{FM}$. □

The role of the Bass modules

[Šaroch]

Let $\mathcal{C}$ be a class of countably presented modules, and $\mathcal{L}$ the class of all locally $\mathcal{C}$-free modules.

Let B be a Bass module over $\mathcal{C}$ such that B is not a direct summand in a module from $\mathcal{L}$.

Then B has no $\mathcal{L}$-precover.

A generalization via tilting theory

T is a (large) tilting module provided that

- T has finite projective dimension,
- $\mathrm{Ext}^i_R(T, T^{(\kappa)}) = 0$ for each cardinal κ, and
- there exists an exact sequence $0 \to R \to T_0 \to \cdots \to T_r \to 0$ such that $r < \omega$, and for each $i < r$, $T_i \in \mathrm{Add}(T)$, i.e., T_i is a direct summand of a (possibly infinite) direct sum of copies of T .

$\mathcal{B} = \{T\}^{\perp_\infty} = \bigcap_{1<i} \mathrm{KerExt}^i_R(T, -)$ the right tilting class of T.
$\mathcal{A} = {}^\perp\mathcal{B}$ the left tilting class of T.

- $\mathcal{A} \cap \mathcal{B} = \mathrm{Add}(T)$.
- Right tilting classes coincide with the classes of finite type, that is, they have the form $\mathcal{S}^\perp$ where $\mathcal{S}$ is a set of strongly finitely presented modules of bounded projective dimension.
- $\mathcal{A} = \mathrm{Filt}(\mathcal{A}^{\leq\omega})$, hence $\mathcal{A}$ is precovering. Moreover, $\mathcal{A} \subseteq \varinjlim \mathcal{A}^{<\omega}$.

Σ-pure split tilting modules

A module M is Σ-pure split provided that each pure embedding $N' \hookrightarrow N$ with $N \in \mathrm{Add}(M)$ splits.

A tilting module T is Σ-pure split, iff $\mathcal{A} = \varinjlim \mathcal{A}^{<\omega}$, iff $\mathcal{A}$ closed under direct limits.

Examples

Let $T = R$. Then T is a tilting module of projective dimension 0, and T is Σ-pure split iff R is a right perfect ring.

Each Σ-pure injective tilting module is Σ-pure split.
Each finitely generated tilting module over any artin algebra is Σ-pure injective.

Locally T-free modules

Let R be a ring and T a tilting module.

A module M is locally T-free provided that M possesses a set $\mathcal{H}$ of submodules such that

- $\mathcal{H} \subseteq \mathcal{A}^{\leq\omega}$,
- each countable subset of M is contained in an element of $\mathcal{H}$,
- $\mathcal{H}$ is closed under unions of countable chains.

Let $\mathcal{L}$ denote the class of all locally T-free modules.

Note: If M is countably generated, then M is locally T-free, iff $M \in \mathcal{A}^{\leq\omega}$.

Locally T-free modules

For any ring R and any tilting module T, we have

$$\mathcal{A} \subseteq \mathcal{L} \subseteq \varinjlim \mathcal{A}^{<\omega}.$$

Example

Let R be an arbitrary ring and $T = R$. Then

$$\mathcal{A} = \mathcal{P}_0 \subseteq \mathcal{L} = \mathcal{FM} \subseteq \varinjlim \mathcal{A}^{<\omega} = \mathcal{F}_0.$$

Locally T-free modules and approximations

Theorem

Let R be a ring and T be a tilting module. Then TFAE:

1. *$\mathcal{L}$ is (pre)covering.*
2. *$\mathcal{L}$ is deconstructible.*
3. *T is Σ-pure split.*

Note: The theorem on flat Mittag-Leffler modules stated earlier is just the particular case of $T = R$.

The role of Bass modules, and Enochs' Conjecture

Theorem

$\mathcal{L}$ is (pre)covering, iff $\mathcal{A}$ is closed under direct limits, iff $B \in \mathcal{A}$ for each Bass module B over $\mathcal{A}^{<\omega}$ (i.e., $\varinjlim_{\omega}(\mathcal{A}^{<\omega}) \subseteq \mathcal{A}$).

Enochs' Conjecture

Let $\mathcal{C}$ be a class of modules. Then $\mathcal{C}$ is covering, iff $\mathcal{C}$ is precovering and closed under direct limits.

Corollary

Enochs' Conjecture holds for all left tilting classes of modules.

A finite dimensional example

Let R be an indecomposable hereditary finite dimensional algebra of infinite representation type.
Then there is a partition of ind-R into three sets:

q ... the indecomposable preinjective modules

p ... the indecomposable preprojective modules

t ... the regular modules (the rest).

Then $p^\perp$ is a right tilting class (and $M \in p^\perp$, iff M has no non-zero direct summands from p).

The tilting module T inducing $p^\perp$ is called the Lukas tilting module.
The left tilting class of T is the class of all Baer modules.
The locally T-free modules are called locally Baer modules.

Non-precovering classes of locally Baer modules

Theorem

- *The class of all Baer modules coincides with* $Filt(p)$.
- *The Lukas tilting module* T *is countably generated, but has no finite dimensional direct summands, and it is not* Σ*-pure split. So* $\mathcal{L}$ *is not precovering (and hence not deconstructible).*

The Bass modules behind the scene

The relevant Bass modules can be obtained as unions of the chains

$$P_0 \overset{f_0}{\hookrightarrow} P_1 \overset{f_1}{\hookrightarrow} \dots \overset{f_{i-1}}{\hookrightarrow} P_i \overset{f_i}{\hookrightarrow} P_{i+1} \overset{f_{i+1}}{\hookrightarrow} \dots$$

such that all the P_i are preprojective (i.e., in $\mathrm{add}(p)$), but the cokernels of all the f_i are regular (i.e., in $\mathrm{add}(t)$).

Almost split maps and sequences

Definition

Let R be a ring and N a non-projective module. An epimorphism of modules $f : M \to N$ is right almost split, if f is not split, and if $g : P \to N$ is not a split epimorphism in $\mathrm{Mod}\text{–}R$, then g factorizes through f.

Dually, we define a left almost split monomorphism $f' : N' \to M'$ for N' non-injective.

A short exact sequence of modules $0 \to N' \stackrel{f'}{\to} M \stackrel{f}{\to} N \to 0$ is almost split, if it does not split, f is a right almost split epimorphism, and f' is a left almost split monomorphism.

Theorem (Auslander)

Let N is an (indecomposable) finitely presented non-projective module with local endomorphism ring. Then there always exists a right almost split epimorphism $f : M \to N$.

Auslander's problem and generalized tree modules

Auslander'1975

Are there further cases where a right almost split epimorphism ending in a non-projective module N exists?

A negative answer has recently been given using (generalized) tree modules:

Theorem (Šaroch'2015)

Let R be a ring and N be a non-projective module. TFAE:

1. *Then there exists a right almost split epimorphism $f : M \longrightarrow N$.*
2. *N is finitely presented and its endomorphism ring is local.*

Corollary

Let R be a ring and $0 \to N' \to M \to N \to 0$ an almost split sequence in Mod-R. Then N is finitely presented with local endomorphism ring, and N' is pure-injective.

References

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