Multiplicative bases for infinite dimensional algebras

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I. Multiplicative bases and their generalizations

Measuring complexity of multiplication in an algebra in terms of its linear structure

Let K be a field, R be (an associative unital) K-algebra, and B be a K-linear space basis of R.

Each $r \in R$ is uniquely a K-linear combination of elements of B: $r = \sum_{b \in B} b k_b$ where $k_b = 0$ for almost all $b \in B$.

$\mathrm{supp}(r) = \{b \in B \mid k_b \neq 0\}$ and $\mathrm{cs}(r) = \mathrm{card}\,(\mathrm{supp}(r))$.

If $1 \leq k < \omega$, then the basis B is k-bounded, if $\mathrm{cs}(b.b') \leq k$ for all $b, b' \in B$. Equivalently, $\mathrm{cs}(r.r') \leq k.\mathrm{cs}(r).\mathrm{cs}(r')$ for all $r, r' \in R$.

For example, if $\dim R_K = d < \infty$, then each K-basis B of R is d-bounded.

A 1-bounded basis is also called a weak multiplicative basis.

B is a bounded basis of R, if B is k-bounded for some $1 \leq k < \omega$.

Multiplicative bases

B is a multiplicative basis of R, if either $bb' = 0$ or $bb' \in B$, for all $b, b' \in B$.

B is a strong multiplicative basis of R, if $bb' \in B$ for all $b, b' \in B$.

strong multiplicative $\Longrightarrow$ **multiplicative** $\Longrightarrow$ **weak multiplicative** $\Longrightarrow$ k**-bounded** $\Longrightarrow$ l**-bounded** $\Longrightarrow$ **bounded** $(1 \leq k \leq l < \omega)$.

Some cases of coincidence: If $K = 2$, then weak multiplicative $=$ multiplicative. If R has no zero divisors, then each multiplicative basis is strong.

The point

If B is a multiplicative basis of a K-algebra R then the K-algebra structure of R is completely determined by the multiplicative semigroup $(B \cup \{0\}, .)$.

The ubiquity of multiplicative bases

Polynomial algebras

The multiplicative K-basis consisting of all monic monomials is strong.

Group algebras and their generalizations

Let G be any group. Then G is a strong multiplicative basis of the group algebra KG. More in general, if M is a monoid then M is a strong multiplicative basis of the monoid algebra KM.

If S is a semigroup such that the semigroup algebra $R = KS$ is unital, then S is a strong multiplicative basis of the K-algebra R. Conversely: if B is a strong multiplicative basis of a (unital) K-algebra R, then R is isomorphic to the semigroup algebra KB.

A particular feature of semigroup algebras

$\Psi : \sum_{s \in S} sk_s \mapsto \sum_{s \in S} k_s$ is a (unital) K-algebra homomorphism whose kernel (called the fundamental ideal of KS) has codimension 1.

More examples

Full matrix algebras

For each $1 \leq n < \omega$, the algebra $M_n(K)$ over a field K has a multiplicative basis B consisting of the matrix units e_{ij} $(i, j < n)$.
If $n \geq 2$, then $M_n(K)$ has no strong multiplicative basis (as $M_n(K)$ is a simple ring).

Path algebras of quivers

The standard basis B of KQ is multiplicative. If Q has no oriented cycles, then KQ has a strong multiplicative basis.

[Gabriel] Any finite dimensional indecomposable basic **hereditary** algebra over an algebraically closed field has a multiplicative basis.
[Bautista-Gabriel-Roiter-Salmeron] Any finite dimensional algebra **of finite representation type** over any algebraically closed field K has a multiplicative basis. However, this fails when K is not algebraically closed.

The case of skew-fields

- $\{1, i\}$ is a weak multiplicative $\mathbb{R}$-basis of the field $\mathbb{C}$.
- $\{1, i, j, k\}$ is a weak multiplicative $\mathbb{R}$-basis of the quaternion algebra $\mathbb{H}$.

Do $\mathbb{C}$ or $\mathbb{H}$ have a multiplicative $\mathbb{R}$-basis?

Lemma

Let K be field and $K \subsetneq K'$ be a skew-field extension. Then K' has no multiplicative K-basis.

Corollary

If the field K is not algebraically closed than K has a finite field extension $K \subsetneq K'$ which is of FRT, but has no multiplicative basis.

Induced multiplicative bases

Green's Lemma

Let R be a K-algebra with a multiplicative basis B, and I be an ideal in R. Let $C = \{b + I \mid b \notin I\}$.
Then C is a multiplicative K-basis of the algebra R/I, iff I is 2-nomial w.r.t. B.

I 2-nomial w.r.t. B, if I is generated by (some) elements of the form $b_1 - b_2$ and b, where $b, b_1, b_2 \in B$.

C is the induced multiplicative basis on R/I.

From multiplicative to Gröbner bases

The case of polynomials

$R = k[x_1, ..., x_n]$, $B =$ all monic monomials in R. A well-order $>$ on B is admissible if $1 \leq b$ for all $b \in B$, and if $b < b'$ and $c \in B$, then $bc < b'c$.

For each $0 \neq f \in R$, let $\ell m(f)$ be the $>$-largest element of B occurring in f. For a subset $S \subseteq R$, let $in(S)$ be the initial ideal of S, i.e., the ideal of R generated by $\{\ell m(f) \mid f \in S\}$.

[Dickson]

Each ideal $0 \neq I$ of R contains a minimal finite subset $G \subseteq I$ such that $in(G) = in(I)$. G is the Gröbner basis of I.

An application [Macaulay]

Let I be an ideal of R, G be its Gröbner basis, and $B' = \{b \in B \mid b \neq \ell m(g) \text{ for all } g \in G\}$. Then $B' + I$ is a K-basis of R/I.

From multiplicative to Gröbner bases

General Gröbner bases

A well-order $>$ on a multiplicative basis B of a (non-commutative) K-algebra R is admissible in case

- for all $b_1, b_2, b_3 \in B$, if $b_1 > b_2$ then $b_1 b_3 > b_2 b_3$ if both $b_1 b_3$ and $b_2 b_3$ are nonzero.
- for all $b_1, b_2, b_3 \in B$, if $b_1 > b_2$ then $b_3 b_1 > b_3 b_2$ if both $b_3 b_1$ and $b_3 b_2$ are nonzero.
- for all $b_1, b_2, b_3, b_4 \in B$, if $b_1 = b_2 b_3 b_4$ then $b_1 \geq b_3$.

The admissible order induces the notion of a leading term (or tip) of each $0 \neq r \in R$. For a subset $S \subseteq R$, let $t(S)$ be the ideal generated by the tips of all $0 \neq s \in S$.

Let I be an ideal of R. Then a subset G of I is a Gröbner basis of I in case $t(G) = t(I)$.

The ubiquity of path algebras

[Green]

Let R be a K-algebra possessing a multiplicative basis with an admissible well-order. Then R has a Gröbner basis.

For example, the standard basis of a path algebra KQ of a quiver Q with a finite set of vertices admits an admissible well-order.

[Green]

Let R be a K-algebra with a multiplicative basis B equipped with an admissible order $>$.
Then there exists a quiver Q with a finite set of vertices, and an isomorphism $KQ/I \overset{\varphi}{\cong} R$ where I is some 2-nomial ideal of the path algebra KQ equipped with the standard basis S.
Moreover, $B = \{\varphi(s + I) \mid s \notin I\}$.

Normed bases for finite representation type

[Bautista-Gabriel-Roiter-Salmerón]

Let K be an algebraically closed field and R be a finite dimensional K-algebra. Assume R is of finite representation type. Then R has a normed multiplicative K-basis B.

B is normed in case B contains a complete set of primitive orthogonal idempotents of R, as well as a K-basis of each power of the Jacobson radical of R.

A point of the proof: The standard basis of any path algebra is normed, so by Green's Lemma, it suffices to show that $R \cong KQ/I$ for a quiver Q and a 2-nomial ideal I of the path algebra KQ, and use the induced basis.

Corollary

Let K be an algebraically closed field. Then for each $n \geq 1$, there are only finitely many K-algebras of finite representation type of dimension n.

II. Transfinite extensions of simple artinian rings

Semiartinian regular rings

- A ring R is right semiartinian, if R is the last term of the right Loewy sequence of R, i.e., there are an ordinal σ and a strictly increasing sequence $(S_\alpha \mid \alpha \leq \sigma + 1)$, such that $S_0 = 0$, $S_{\alpha+1}/S_\alpha = \mathrm{Soc}(R/S_\alpha)$ for all $\alpha \leq \sigma$, $S_\alpha = \bigcup_{\beta<\alpha} S_\beta$ for all limit ordinals $\alpha \leq \sigma$, and $S_{\sigma+1} = R$.
- R is (von Neumann) regular, if all (right R-) modules are flat.
- R has right primitive factors artinian (has right pfa for short) in case R/P is right artinian for each right primitive ideal P of R.

Let R be a regular ring.

- R is right semiartinian, iff it is left semiartinian, and the right and left Loewy sequences of R coincide.
- R has right pfa, iff it has left pfa, iff all homogenous completely reducible (left or right) modules are injective.
- If R is commutative, then R has pfa.

Structure of semiartinian regular rings with pfa

Let R be a right semiartinian ring and $(S_\alpha \mid \alpha \leq \sigma + 1)$ be the right Loewy sequence of R with $\sigma \geq 1$. The following conditions are equivalent:

- R is regular with pfa.

- for each $\alpha \leq \sigma$ there are a cardinal λ_α, positive integers $n_{\alpha\beta}$ $(\beta < \lambda_\alpha)$ and skew-fields $K_{\alpha\beta}$ $(\beta < \lambda_\alpha)$ such that $S_{\alpha+1}/S_\alpha \cong \bigoplus_{\beta<\lambda_\alpha} M_{n_{\alpha\beta}}(K_{\alpha\beta})$, as rings without unit. Moreover, λ_α is infinite iff $\alpha < \sigma$.
  The pre-image of $M_{n_{\alpha\beta}}(K_{\alpha\beta})$ in this isomorphism coincides with the βth homogenous component of $\mathrm{Soc}(R/S_\alpha)$, and it is finitely generated as right R/S_α-module for all $\beta < \lambda_\alpha$.

$P_{\alpha\beta}$:= a representative of simple modules in the βth homogenous component of $S_{\alpha+1}/S_\alpha$. $Zg(R) := \{P_{\alpha\beta} \mid \alpha \leq \sigma, \beta < \lambda_\alpha\}$ is a set of representatives of all simple modules, and also the Ziegler spectrum of R. The Cantor-Bendixson rank of $Zg(R)$ is σ.

Transfinite extensions of simple artinian rings

R/S_σ	$M_{n_{\sigma 0}}(K_{\sigma 0}) \oplus$	...	$\oplus M_{n_{\sigma,\lambda_\sigma - 1}}(K_{\sigma,\lambda_\sigma - 1})$		
...	...	...	...		
$S_{\alpha+1}/S_\alpha$	$M_{n_{\alpha 0}}(K_{\alpha 0}) \oplus$	...	$\oplus M_{n_{\alpha\beta}}(K_{\alpha\beta}) \oplus$ ...		$\beta < \lambda_\alpha$
...	...	...	...		
S_2/S_1	$M_{n_{10}}(K_{10}) \oplus$	...	$\oplus M_{n_{1\beta}}(K_{1\beta}) \oplus$ ...		$\beta < \lambda_1$
$S_1 = Soc(R)$	$M_{n_{00}}(K_{00}) \oplus$	...	$\oplus M_{n_{0\beta}}(K_{0\beta}) \oplus$ ...		$\beta < \lambda_0$

$\sigma + 1 =$ **Loewy length** of R (at least 2).

$\lambda_\alpha =$ **number of homogenous components** of the αth layer of R $(\alpha \leq \sigma)$. (Infinite except for $\alpha = \sigma$).

$n_{\alpha\beta} =$ (finite) **dimension** of βth homogenous component in αth layer.

$K_{\alpha\beta} =$ **endomorphism skew-field** of a simple module in βth homogenous component of the αth layer $(\alpha \leq \sigma,\ \beta < \lambda_\alpha)$.

The dimension sequence

The sequence $\mathcal{D}_R = \{(\lambda_\alpha, \{(n_{\alpha\beta}, K_{\alpha\beta}) \mid \beta < \lambda_\alpha\}) \mid \alpha \leq \sigma\}$ is an invariant of R. It is called the dimension sequence of R.

The hereditary case

If the Loewy length σ of R is countable and the αth layer of the socle sequence of R is countably generated for each $0 < \alpha < \sigma$, then R is hereditary. So R is always hereditary in case it has Loewy length 2.

The simplest example

The K-algebra of all eventually constant sequences in K^ω over a field K.

Multiplicative bases and selfinjective regular algebras

Examples

The commutative K-algebra $R = K^\kappa$ is regular and self-injective. If κ is finite, then R has a strong multiplicative basis, otherwise R has no bounded basis.

The endomorphism algebra R of a right K-linear space L is regular and right self-injective. If L is finite-dimensional, then R has a multiplicative basis, otherwise R has no bounded basis.

Theorem

Let K be a field and R be a K-algebra which is regular and left or right self-injective, but not completely reducible. Then R has no bounded multiplicative basis.

A multiplicative basis B of a semiartinian K-algebra R is conormed if B contains a complete set of orthogonal idempotents of R, as well as a basis of S_α for each $\alpha \leq \sigma$.

Commutative semiartinian regular algebras

Let K be a field. We will consider the commutative semiartinian regular K-algebras R of the following form:

$S_{\sigma+1}/S_\sigma \cong K^{(\lambda_\sigma)}$
$\dots$
$S_{\alpha+1}/S_\alpha \cong K^{(\lambda_\alpha)}$
$\dots$
$S_2/S_1 \cong K^{(\lambda_1)}$
$S_1/S_0 = Soc(R) \cong K^{(\lambda_0)}$

$K \oplus$	$\dots$	$\oplus K$	
$\dots$	$\dots$	$\dots$	
$K \oplus$	$\dots$	$\oplus K \oplus \dots$	
$\dots$	$\dots$	$\dots$	
$K \oplus$	$\dots$	$\oplus K \oplus \dots$	
$K \oplus$	$\dots$	$\oplus K \oplus \dots$	

$\mathcal{D}_R = (\lambda_\alpha \mid \alpha \leq \sigma)$ is the (simplified) dimension sequence of the K-algebra R.

Upto a K-algebra isomorphism, R is a subalgebra of the K-algebra K^{λ_0} with $Soc(R) = K^{(\lambda_0)}$.

The simplified dimension sequence $\mathcal{D}$ is countable if both the ordinal σ and all the cardinals λ_α ($\alpha \leq \sigma$) are countable.

Algebras of countable type and dimension sequences

Let K be a field. A commutative semiartinian regular K-algebra R of the form above is called of countable type, if it dimension sequence $\mathcal{D}_R$ is countable.

Existence Theorem

If $\mathcal{D}$ is a countable dimension sequence, then there exists a K-algebra R of countable type such that $\mathcal{D} = \mathcal{D}_R$.

Uniqueness Theorem

Let R and R' be K-algebras of countable type. Then $R \cong R'$ as K-algebras, iff $\mathcal{D}_R = \mathcal{D}_{R'}$.

A key point of uniqueness: countable layers are connected, as countable sets of pairwise orthogonal idempotents lift modulo any ideal of R. *Uniqueness fails even for $\sigma = 1$ when λ_0 is uncountable, or when all layers are countable, but the length is $\aleph_1$.*

The construction

The induction step

Let K be a field, κ an infinite cardinal, and $\mathcal{R} = (R_\alpha \mid \alpha < \kappa)$ a sequence of K-algebras.

Let $P = \prod_{\alpha<\kappa} R_\alpha$ denote the K-algebra product of the algebras in $\mathcal{R}$. Let $I = \bigoplus_{\alpha<\kappa} R_\alpha$ (so I is an ideal in P).

Let $R = R(\kappa, K, \mathcal{R})$ denote the K-subalgebra of P defined by $R = I \oplus 1_P \cdot K$.

Assume that each K-algebra R_α $(\alpha < \kappa)$ is semiartinian and has a conormed multiplicative basis. Then so does the K-algebra $R(\kappa, K, \mathcal{R})$.

The construction

Definition of the algebras $B_{\alpha,n}$

Let K be a field.

(i) $B_{0,1} = K$.

(ii) For a non-limit ordinal $\alpha = \beta + 1$, we let $\mathcal{R} = (R_m \mid m < \aleph_0)$ be the constant sequence of K-algebras $R_m = B_{\beta,1}$ for each $m < \aleph_0$. We let $B_{\alpha,1} = R(\aleph_0, K, \mathcal{R})$.

(iii) For each limit ordinal α, we put $R_\beta = B_{\beta,1}$ for each $\beta < \alpha$. We let $B_{\alpha,1} := R(\alpha, K, \mathcal{R})$ where $\mathcal{R} = (R_\beta \mid \beta < \alpha)$.

(iv) For all $1 < n < \omega$ and all ordinals α, we let $B_{\alpha,n} = B_{\alpha,1} \boxplus \cdots \boxplus B_{\alpha,1}$ (the direct product of n copies of the K-algebra $B_{\alpha,1}$).

Example: $B_{1,1}$ is the K-algebra of all eventually constant sequences in $K^{\aleph_0}$.

Each of the K-algebras $B_{\alpha,n}$ ($\alpha \geq 0$, $1 \leq n < \omega$) has a conormed multiplicative basis.

Conormed multiplicative bases

Theorem

Let K be a field and $\mathcal{D}$ be a countable dimension sequence of length σ. Then there exists $1 \leq n < \omega$ such that $\mathcal{D}$ is the dimension sequence of the K-algebra $B_{\sigma,n}$.

Corollary

All algebras of countable type have conormed multiplicative bases.

References

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