Locality for qc-sheaves associated with tilting

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In honour of Mike Prest

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Tilting modules and classes

Definition

A (right R-) module T is tilting provided that

(T1) T has finite projective dimension,

(T2) $\mathrm{Ext}^i_R(T, T^{(\kappa)}) = 0$ for each cardinal κ and each $i > 0$, and

(T3) there exists an exact sequence $0 \to R \to T_0 \to \cdots \to T_r \to 0$ such that $r < \omega$, and for each $i \leq r$, $T_i \in \mathrm{Add}(T)$, i.e., T_i is a direct summand in a direct sum of copies of T .

If $\mathrm{pd}_R(T) \leq n$, then T is called n-tilting.
0-tilting modules = projective generators (possibly infinitely generated).

$\mathcal{B} = \bigcap_{i>0} \mathrm{KerExt}^i_R(T, -)$ is the (right) tilting class induced by T.

$\mathcal{A} = \mathrm{KerExt}^1_R(-, \mathcal{B})$ is the left tilting class induced by T.

A tilting module $\tilde{T}$ is equivalent to T in case $\mathrm{Add}(T) = \mathrm{Add}(\tilde{T})$.

Basic restriction for the commutative setting

Let R be a commutative ring.

(i) If T is a tilting module of projective dimension > 0, then T is not finitely generated.

(ii) If $T \in \mathrm{mod}\text{–}R$ and $1 \leq \mathrm{pd}_R(T) = n < \infty$, then $\mathrm{Ext}^n_R(T, T) \neq 0$. Hence T fails condition (T2).

Proof of (ii)

All syzygies of T are finitely generated, so $\mathrm{pd}_R(T) = \max_m \mathrm{pd}_{R_m}(T_m)$. Take $m \in \mathrm{mSpec}(R)$ such that $\mathrm{pd}_{R_m}(T_m) = n$. Then $T_m \in \mathrm{mod}\text{–}R_m$ and $(\mathrm{Ext}^n_R(T, T))_m \cong \mathrm{Ext}^n_{R_m}(T_m, T_m)$, so w.l.o.g., R is local. Let $\mathcal{F}$ be the minimal free resolution of T. $\mathcal{F}$ is given by an iteration of projective covers, so $d_k(F_k) \subseteq mF_{k-1}$ for each $k > 0$ where d_k is the differential. As $\mathrm{Ext}^n_R(T, -)$ is right exact, the epimorphism $T \to T/mT$ induces a surjection $\mathrm{Ext}^n_R(T, T) \to \mathrm{Ext}^n_R(T, T/mT)$. However, $\mathrm{Ext}^n_R(T, T/mT) \cong \mathrm{Ext}^1_R(\Omega^{n-1}(T), T/mT) \cong \mathrm{Hom}_R(F_n, T/mT) \neq 0$. $\square$

Tilting abelian groups

Theorem

Let $P \subseteq \mathrm{Spec}(\mathbb{Z}) \setminus \{0\}$*, and* $\mathbb{Z} \subseteq A_P \subseteq \mathbb{Q}$ *and* $A_P/\mathbb{Z} \cong \bigoplus_{p \in P} \mathbb{Z}_{p^\infty}$.

- $T_P = A_P \oplus A_P/\mathbb{Z}$ *is a tilting abelian group,*
- $\mathcal{B}_P = \{A \in \mathrm{Mod}\text{–}\mathbb{Z} \mid Ap = A \text{ for all } p \in P\}$.

Each tilting abelian group is equivalent to T_P *for some* $P \subseteq \mathrm{Spec}(\mathbb{Z}) \setminus \{0\}$.

Finite type

Theorem

Let T be a tilting module with the induced left and right tilting classes $\mathcal{A}$ and $\mathcal{B}$, respectively.

Then there is a set $\mathcal{S}$ consisting of strongly finitely presented modules in $\mathcal{A}$, such that

$$\mathcal{B} = KerExt^1_R(\mathcal{S}, -).$$

One can always take $\mathcal{S} = \mathcal{A} \cap \mathrm{mod}\text{–}R$; in this case $\mathcal{A} \subseteq \varinjlim \mathcal{S}$.

Terminology: The set $\mathcal{S}$ witnesses the finite type of T.

Abelian groups, cont.

For $P \subseteq \mathrm{Spec}(\mathbb{Z}) \setminus \{0\}$, we can take $\mathcal{S}_P = \{\mathbb{Z}_p \mid p \in P\}$.

Characteristic sequences over commutative rings

- A subset $P \subseteq \mathrm{Spec}(R)$ is Thomason, if $P = \bigcup_{I \in \mathcal{I}} V(I)$ for a set $\mathcal{I}$ consisting of finitely generated ideals of R.
  Here, $V(I) = \{p \in \mathrm{Spec}(R) \mid I \subseteq p\}$.
- If $M \in \mathrm{Mod}\text{–}R$ and $p \in \mathrm{Spec}(R)$, then p is vaguely associated to M, if R/p is contained in the smallest subclass of $\mathrm{Mod}\text{–}R$ containing M and closed under submodules and direct limits. Def.: $p \in \mathrm{Vass}(M)$.

Example: If R is noetherian, then Thomason subsets = upper subsets, and weakly associated primes = associated primes.

Let R be a commutative ring. A sequence $\bar{P} = (P_0, \dots, P_{n-1})$ of Thomason subsets of $\mathrm{Spec}(R)$ is called characteristic, if

- $P_0 \supseteq P_1 \supseteq \cdots \supseteq P_{n-1}$, and
- $\mathrm{Vass}(\Omega^{-i}(R)) \cap P_i = \emptyset$ for all $i < n$.

Structure of tilting classes over commutative rings

For R commutative and $n \geq 1$, there is a bijection between:

- characteristic sequences of length n, and
- right n–tilting classes in $\mathrm{Mod}\text{–}R$.

The right n-tilting class $\mathcal{T}_P$ corresponding to a characteristic sequence $\bar{P} = (P_0, \dots, P_{n-1})$, where $P_i = \bigcup_{I \in \mathcal{I}_i} V(I)$ for each $i < n$, equals

$$\mathcal{T}_{\bar{P}} = \{M \in \mathrm{Mod}\text{–}R \mid \mathrm{Tor}_i^R(R/I, M) = 0 \;\; \forall i < n \; \forall I \in \mathcal{I}_i\}.$$

For an n-tilting module T inducing the n-tilting class $\mathcal{T}$, the corresponding characteristic sequence $\bar{P}_T = (P_0, \dots, P_{n-1})$ satisfies for each $i < n$

$$P_i = \{p \in \mathrm{Spec}(R) \mid \exists i < k \leq n : \; \mathrm{Tor}_k^R(E(R/p), T) \neq 0\}.$$

(Note: the structure of tilting modules is still an open problem ...)

Quasi-coherent sheaves as representations

Let X be a scheme and $\mathcal{R} = (\mathcal{R}(U) \mid U \subseteq X, U \text{ open affine })$ be its structure sheaf.

A quasi-coherent sheaf $\mathcal{M}$ on X can be represented by an assignment

- to every affine open subset $U \subseteq X$, an $\mathcal{R}(U)$-module $\mathcal{M}(U)$ of sections, and
- to each pair of open affine subsets $V \subseteq U \subseteq X$, an $\mathcal{R}(U)$-homomorphism $f_{UV} : \mathcal{M}(U) \to \mathcal{M}(V)$ such that

$$\mathrm{id}_{\mathcal{R}(V)} \otimes f_{UV} : \mathcal{R}(V) \otimes_{\mathcal{R}(U)} \mathcal{M}(U) \to \mathcal{R}(V) \otimes_{\mathcal{R}(U)} \mathcal{M}(V) \cong \mathcal{M}(V)$$

is an $\mathcal{R}(V)$-isomorphism.

$+$ compatibility conditions: if $W \subseteq V \subseteq U$, then $f_{UV} f_{VW} = f_{UW}$.

Properties of the representations

Exactness

The functors $\mathcal{R}(V) \otimes_{\mathcal{R}(U)} -$ are exact, i.e., the $\mathcal{R}(U)$-modules $\mathcal{R}(V)$ are flat (in fact, they are "very flat").

Non-uniqueness of the representations

Not all affine open subsets are needed: a set of them, $\mathcal{S}$, covering both X, and all $U \cap V$ where $U, V \in \mathcal{S}$, will do.

The affine case (Grothendieck)

If $X = \mathrm{Spec}(R)$ for a commutative ring R, then $\mathcal{S} = \{X\}$ is enough, so

quasi-coherent sheaves on X = R-modules.

Extending properties from modules to qc-sheaves

Definition

Let $\mathfrak{P}$ be a property of modules. A qc-sheaf $\mathcal{M}$ on a scheme X is a locally $\mathfrak{P}$ qc-sheaf provided that for **each** open affine set U of X, $\mathcal{M}(U)$ satisfies $\mathfrak{P}$ as an $\mathcal{R}(U)$-module.

Requirement: Properties studied for qc-sheaves should be "independent of coordinates", i.e., for each scheme X, it should be possible to test for the property using an (arbitrary) open affine covering of X.

Definition

The notion of a locally $\mathfrak{P}$ qc-sheaf is Zariski local if for each scheme X, each open affine covering $X = \bigcup_{V \in \mathcal{S}} V$ of X, and each qc-sheaf $\mathcal{M}$ on X, if $\mathcal{M}(V)$ satisfies $\mathfrak{P}$ as $\mathcal{R}(V)$-module for each $V \in \mathcal{S}$, then $\mathcal{M}$ is a locally $\mathfrak{P}$-qc-sheaf.

Ascent and descent along flat base changes

Definition

A property of modules $\mathfrak{P}$ is called an ad-property provided that

- if $\varphi : R \to S$ is any flat ring homomorphism, then $\mathfrak{P}$ ascends along φ (i.e., if M satisfies $\mathfrak{P}$ as R–module, then so does $M \otimes_R S$ as S–module), and
- if $\varphi : R \to S$ is any faithfully flat ring monomorphism, then $\mathfrak{P}$ descends along φ (i.e., if $M \otimes_R S$ satisfies $\mathfrak{P}$ as S-module, then so does M as R–module).

Sufficient conditions for Zariski locality

Lemma

Let $\mathfrak{P}$ be an ad-property of modules over commutative rings.
Then the notion of a locally $\mathfrak{P}$ qc-sheaf is Zariski local.

Affine Communication Lemma (ACL)

A weaker property is sufficient:

(1) $\mathfrak{P}$ ascends along all localizations $\varphi : R \to R_f$ where $f \in R$, and
(2) $\mathfrak{P}$ descends along all faithfully flat ring monomorphisms of the form $\varphi_{f_0,\dots,f_{m-1}} : R \to \prod_{j<m} R_{f_j}$ where $R = \sum_{j<m} f_j R$.

The basic example: vector bundles

Definition

A qc-sheaf $\mathcal{M}$ on a scheme X is an (infinite dimensional) vector bundle, if $\mathcal{M}(U)$ is a projective $\mathcal{R}(U)$-module for each open affine set U of X.

(So vector bundles are exactly the locally projective qc-sheaves.)

Theorem

The notion of a vector bundle is Zariski local.

(Conjectured by Grothendieck in the 1960's, proved in 1971 by Raynaud and Gruson, by showing that projectivity is an ad-property.)

Locally tilting quasi-coherent sheaves

Definition

Let $n < \omega$. A qc-sheaf $\mathcal{M}$ on a scheme X is locally n-tilting, if $\mathcal{M}(U)$ is an n-tilting $\mathcal{R}(U)$-module for each open affine set U of X.

Let $0 \leq n$. A qc-sheaf $\mathcal{M}$ on a scheme X is locally left n-tilting (locally Add-n-tilting), if for each open affine set U of X, there exists an n-tilting $\mathcal{R}(U)$-module $\mathcal{T}(U)$ such that $\mathcal{M}(U) \in \mathcal{A}_{T(U)}$ ($\mathcal{M}(U) \in \mathrm{Add}(T(U))$).

Problem: Are these three notions Zariski local for each n?

(The latter two are Zariski local for $n = 0$ by the Theorem of Raynaud and Gruson above, as they coincide with the notion of a vector bundle.)

Tilting and flat base change

Ascent-Descent Lemma

Let $\varphi : R \to S$ be a flat ring homomorphism of commutative rings, and T be an n-tilting R-module with the left and right tilting classes $\mathcal{A}$ and $\mathcal{B}$, respectively.

- $T' = T \otimes_R S$ is an n-tilting S-module. So the property of being an n-tilting module ascends along φ.
- Let $\mathcal{B}'$ be the right n-tilting class induced by T'. Then $\mathcal{B}' = \mathcal{B} \cap \mathrm{Mod}\text{–}S$, and for each module $N \in \mathrm{Mod}\text{–}R$, $N \in \mathcal{B}$, iff $N \otimes_R S \in \mathcal{B}'$.
- If φ is a faithfully flat ring monomorphism, then for each $M \in \mathrm{Mod}\text{–}R$, $M \in \mathcal{A}$, iff $M \otimes_R S \in \mathcal{A}'$, where $\mathcal{A}'$ is the left n-tilting class induced by T'.

Tools: Mittag-Leffler conditions

Definition

An inverse system of modules $\mathcal{H} = (H_i, h_{ij} \mid i \leq j \in I)$ is Mittag-Leffler, if for each $k \in I$ there exists $k \leq j \in I$, such that $\mathrm{Im}(h_{kj}) = \mathrm{Im}(h_{ki})$ for each $j \leq i \in I$, that is, the terms of the decreasing chain $(\mathrm{Im}(h_{ki}) \mid k \leq i \in I)$ of submodules of H_k stabilize.

Let B be a module and $\mathcal{M} = (M_i, f_{ji} \mid i \leq j \in I)$ a direct system of finitely presented modules. An application of $\mathrm{Hom}_R(-, B)$ yields the induced inverse system $\mathcal{H} = (H_i, h_{ij} \mid i \leq j \in I)$, where $H_i = \mathrm{Hom}_R(M_i, B)$ and $h_{ij} = \mathrm{Hom}_R(f_{ji}, B)$ for all $i \leq j \in I$.

Let $\mathcal{B}$ be a class of modules. A module M is $\mathcal{B}$-stationary, if M is a direct limit of a direct system $\mathcal{M}$ of finitely presented modules so that for each $B \in \mathcal{B}$, the induced inverse system $\mathcal{H}$ is Mittag-Leffler.

Tools: Mittag-Leffler conditions and generalized dévisage

Lemma

In the setting of the Ascent-Descent Lemma, let $\mathcal{S} = \mathcal{A} \cap \mathrm{mod}\text{–}R$*. Let* $C \in \mathrm{Mod}\text{–}R$ *and* $C' = C \otimes_R S$*.*

- *If* $C \in \varinjlim \mathcal{S}$ *is countably presented, then* $C \in \mathcal{A}$*, iff* C *is* $\mathcal{B}$*-stationary.*
- *Assume that* φ *is faithfully flat, and* $C' \in \varinjlim(\mathcal{S} \otimes_R S)$ *is countably presented. Then* $C \in \varinjlim \mathcal{S}$ *and* C *is countably presented. Moreover, the equivalent conditions above are further equivalent to* C' *being* $\mathcal{B}'$*-stationary, and hence to* $C' \in \mathcal{A}'$*.*

- *If* φ *is faithfully flat, then for each* $M \in \mathrm{Mod}\text{–}R$*,* $M \in \mathcal{A}$*, iff* $M \otimes_R S \in \mathcal{A}'$*.*

Characteristic sequences and flat base change

Let $\varphi : R \to S$ be the faithfully flat ring monomorphism from condition (2) of the ACL.

- If $\bar{P}$ is a characteristic sequence in $\mathrm{Spec}(R)$ corresponding to an n-tilting module T, then the characteristic sequence in $\mathrm{Spec}(S)$ corresponding to $T' = T \otimes_R S$ is $\bar{Q}_{\bar{P}}$, where for each $i < n$, Q_i is defined by $Q_i = \{q \in \mathrm{Spec}(S) \mid \exists p \in P_i : pS \subseteq q\}$.
- Let $T \in \mathrm{Mod}\text{–}R$ be such that $T' = T \otimes_R S$ is an n-tilting S-module. Then the characteristic sequence $\bar{Q}$ in $\mathrm{Spec}(S)$ corresponding to T' is of the form $\bar{Q}_{\bar{P}}$ for some characteristic sequence $\bar{P}$ in $\mathrm{Spec}(R)$. Hence T satisfies conditions (T1) and (T2).

Zariski locality

Theorem

- *The notions of a locally left n-tilting, locally Add-n-tilting, and locally n-tilting quasi-coherent sheaves are Zariski local for each $n \geq 0$.*
- *If R is noetherian, then for each $n \geq 0$, the property of being an n-tilting module is an ad-property.*

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