Řekneme, že $\{a_n\}_{n=1}^{\infty}$ splňuje BC podmínku, platí-li výrok

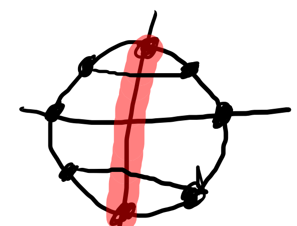
$$\forall \varepsilon > 0\ \exists n_0 \in \mathbb{N}\ \forall n \geq n_0, m \geq n_0 : |a_n - a_m| < \varepsilon.$$

<u>TVRZENÍ</u>: Pokud $\{a_n\}_{n=1}^{\infty}$ nesplňuje BC podmínku, tak $\{a_n\}_{n=1}^{\infty}$ nemá vlastní limitu.

$$(\neg BC \Rightarrow \neg \text{limita}) \Leftrightarrow \underline{(\text{limita} \Rightarrow BC)}$$

<u>Vyšetřete</u>:

1. $\lim\limits_{n\to\infty} \overbrace{\sin\left(\frac{n\pi}{4}\right)}^{a_n}$, $\quad -1 \leq a_n \leq 1 \Rightarrow \lim\limits_{n\to\infty} a_n \neq \infty, -\infty$



$\neg BC: \exists \varepsilon > 0\ \forall n_0 \in \mathbb{N}\ \exists n \geq n_0, m \geq n_0 : |a_n - a_m| \geq \varepsilon$

$\varepsilon = 1$, nechť $n_0 \in \mathbb{N}$. Pozorování: $\sin\left(\frac{8n_0\pi}{4}\right) =$

$= \sin(2\pi n_0) = 0$. Navíc $8n_0 \geq n_0$

Volme $n = 8n_0 + 2$, $m = 8n_0 - 2$.
Potom $n \geq n_0$, $m \geq n_0$ a

$$a_n = \sin\left(\frac{(8n_0+2)\pi}{4}\right) = \sin\left(2\pi n_0 + \frac{1}{2}\pi\right) = 1$$

$$a_m = \sin\left(\frac{(8n_0-2)\pi}{4}\right) = \sin\left(2\pi n_0 - \frac{1}{2}\pi\right) = -1$$

Tedy $|a_n - a_m| = 2 \geq \varepsilon$, tedy $\neg BC$ platí a dle tvrzení neexistuje vlastní limita.

Tvrzení: Nechť $a, b \in \mathbb{R}$, $n \in \mathbb{N}$, pak

$$(a^n - b^n) = (a-b)(a^{n-1} + a^{n-2}b + a^{n-3}b^2 + \dots + ab^{n-2} + b^{n-1})$$

Vzoreček; a, b, $n=3$

2. $\lim\limits_{n\to\infty} \underbrace{\sqrt[3]{n+1}}_{a} - \underbrace{\sqrt[3]{n}}_{b}$

$$= \lim_{n\to\infty} \frac{n+1-n}{(\sqrt[3]{n+1})^2 + \sqrt[3]{n+1}\,\sqrt[3]{n} + (\sqrt[3]{n})^2} =$$

$$= \lim_{n\to\infty} \frac{1}{n^{\frac{2}{3}}(1+\frac{1}{n})^{\frac{2}{3}} + n^{\frac{2}{3}}(1+\frac{1}{n})^{\frac{1}{3}}(1)^{\frac{1}{3}} + n^{\frac{2}{3}}}$$

$$= \lim_{n\to\infty} \underbrace{\frac{1}{n^{\frac{2}{3}}}}_{0} \cdot \frac{1 \to 1}{\underbrace{(1+\frac{1}{n})^{\frac{2}{3}}}_{1} + \underbrace{(1+\frac{1}{n})^{\frac{1}{3}}}_{1} + \underbrace{1}_{1}} \to \frac{1}{3}$$

$$\overset{AL}{=} 0.$$

$$(\sqrt[3]{n+1})^2 = \left(\sqrt[3]{n(1+\tfrac{1}{n})}\right)^2 = (\sqrt[3]{n})^2\left(\sqrt[3]{1+\tfrac{1}{n}}\right)^2 = n^{\frac{2}{3}}(1+\tfrac{1}{n})^{\frac{2}{3}}$$

2'. $$\lim_{n\to\infty} \sqrt{n+1} - \sqrt{n} = \lim_{n\to\infty} \frac{\sqrt{n+1}-\sqrt{n}}{1} \cdot \frac{\sqrt{n+1}+\sqrt{n}}{\sqrt{n+1}+\sqrt{n}}$$

$$= \lim_{n\to\infty} \frac{n+1-n}{\sqrt{n+1}+\sqrt{n}} = \lim_{n\to\infty} \frac{1}{\sqrt{n+1}+\sqrt{n}} \overset{AL}{=} 0$$

6. $\lim\limits_{n\to\infty} \sqrt[n]{n}$, $\quad 1 \le \sqrt[n]{n} \le 1 + d_n$

Definuji: $\sqrt[n]{n} = 1 + d_n \,/\,(\;)^n$ (tj: $d_n = \sqrt[n]{n} - 1$)

$$n = 1 + \binom{n}{1}d_n + \binom{n}{2}d_n^2 + \dots + \binom{n}{n}d_n^n$$

$0 \le d_n$, $\quad n \ge \binom{n}{2}d_n^2 = \frac{n!}{(n-2)!\,2!}d_n^2$

$$= \frac{n(n-1)}{2}d_n^2$$

$$\frac{n}{\frac{n(n+1)}{2}} \ge d_n^2 \iff d_n^2 \le \frac{2}{n+1}$$

$$d_n \le \sqrt{\frac{2}{n+1}}$$

a $\lim\limits_{n\to\infty} \sqrt{\frac{2}{n+1}} = 0$.

Tedy z AL: $\lim\limits_{n\to\infty} 1 + d_n = 1$ a tedy

$$1 \le \sqrt[n]{n} \le 1 + d_n \overset{ZP}{\Longrightarrow} \lim_{n\to\infty} \sqrt[n]{n} = 1$$

$$\underset{\substack{\| \\ 3}}{\sqrt[n]{3^n}} \le \sqrt[n]{n^2+2^n+3^n} \le \sqrt[n]{3^n+3^n+3^n} = \sqrt[n]{3}\,\sqrt[n]{3^n} = \underbrace{\underset{\substack{\downarrow \\ 1}}{\sqrt[n]{3}}\;3}_{\substack{\downarrow \\ 3}}$$

$$\sqrt[n]{3^n} \le \sqrt[n]{n^2+2^n+3^n} \le \sqrt[n]{3^n+3^n+3^n} = \sqrt[n]{3\cdot 3^n}$$

$$\sqrt[n]{3^n} = 3$$

$$= \sqrt[n]{3}\cdot 3 \to 3$$

$$n^2 \le 3^n$$

$$n^2 \le 2^n, \quad \boxed{n \ge 4}$$

$$3 \le \sqrt[n]{n^2+2^n+3^n} \le \sqrt[n]{4^n+4^n+4^n}$$

$$= \sqrt[n]{3}\,\sqrt[n]{4^n} = 4\sqrt[n]{3} \to 4$$

$$\sqrt{n}\left(\sqrt{n+1}-\sqrt{n}\right)$$

$$= \sqrt{n}\,\frac{n+1-n}{\sqrt{n+1}+\sqrt{n}} = \overbrace{\frac{\sqrt{n}}{\sqrt{n}}}^{=1}\,\frac{1}{\underbrace{\sqrt{1+\frac{1}{n}}}_{\downarrow\, 1}+\underbrace{\sqrt{1}}_{\downarrow\, 1}} \xrightarrow[n\to\infty]{AL} \frac{1}{2}$$

$$\sqrt{n\left(1+\frac{1}{n}\right)}+\sqrt{n\cdot 1}$$

$$= \sqrt{n}\left(\sqrt{1+\frac{1}{n}}+\sqrt{1}\right)$$

$$\ldots \lim_{n\to\infty} \ldots\ldots = \frac{1}{2}$$

$$\lim_{n\to\infty} \underbrace{\sqrt{n}\left(\sqrt{n+1}-\sqrt{n}\right)}_{a_n} = \frac{1}{2}$$

$$\Rightarrow \lim_{n\to\infty} -a_n = -\frac{1}{2}$$

$$\lim_{n\to\infty} +a_n = \frac{1}{2}$$

$$a_{2n} \to \frac{1}{2}$$

$$a_{2n+1} \to -\frac{1}{2}$$

Pokud $\lim a_n = A$, $\lim b_n = B$, $A, B \in \mathbb{R}$

$$\Rightarrow \lim a_n b_n = A \cdot B$$

Pokud $\lim a_n = A$, $\lim b_n$ neexistuje

$\not\Rightarrow$ $\lim a_n b_n$ neexistuje

$a_n = \frac{1}{n}$, $A = 0$, $b_n = (-1)^n$, $\lim a_n b_n = 0$.