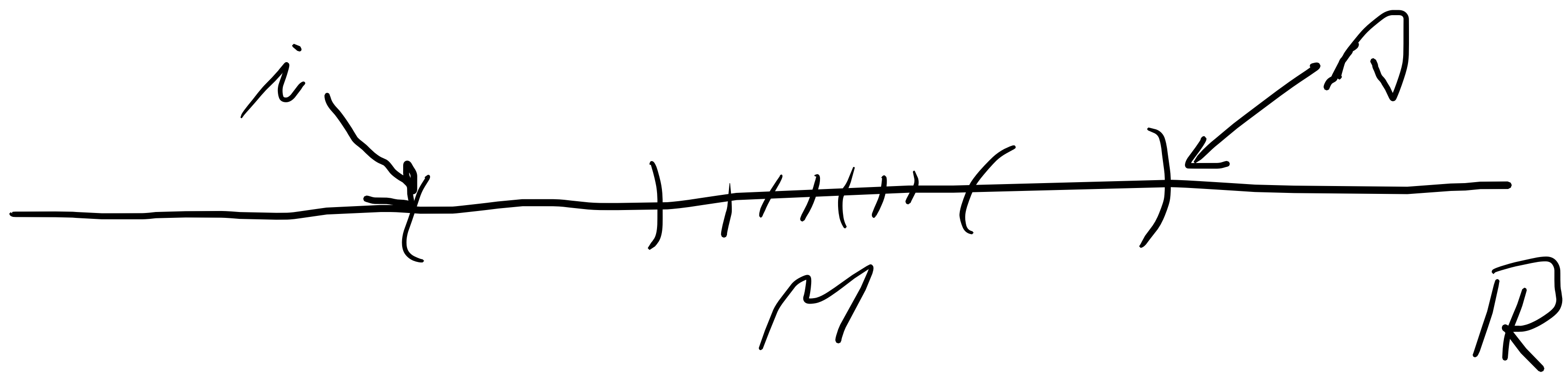
$$xy \le \frac{x^2}{2} + \frac{y^2}{2}$$

<u>Důkaz</u>: $0 \le (x-y)^2 = x^2 - 2xy + y^2$

$$xy \le \frac{x^2+y^2}{2}$$

Nechť $M \subset \mathbb{R}$, řekneme, že $s \in \mathbb{R}$ je supremem množiny M, pokud

1. $\forall x \in M: x \leq s$
2. $\forall \varepsilon > 0 \; \exists x \in M: x > s - \varepsilon$

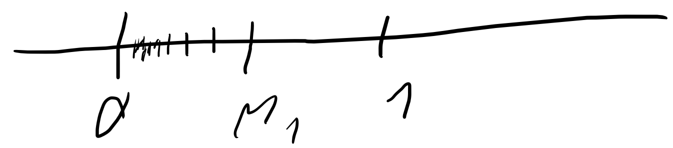


Řekneme, že $i \in \mathbb{R}$ je infimum M, pokud

1. $\forall x \in M: x \geq i$
2. $\forall \varepsilon > 0 \; \exists x \in M: x < s + \varepsilon$

najdete supremum a infimum

$M_1 = \left\{ \frac{1}{n}, n \in \mathbb{N} \right\}$



$\sup M_1 = 1$: 1. $\forall n \in \mathbb{N}: \frac{1}{n} \leq 1$ ✓

2. Je-li $\varepsilon > 0$, volme $n = 1$,

pak $\frac{1}{n} > 1 - \varepsilon$ ✓

$\inf M_1 = 0$: 1. $\forall n \in \mathbb{N}: \frac{1}{n} \geq 0$

2.

Volme $\varepsilon > 0$, nechť $n = \left[\frac{1}{\varepsilon}\right] + 1$,

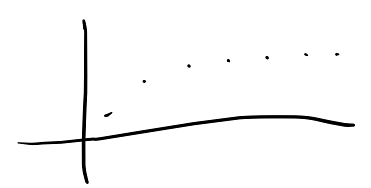
$\left(\begin{array}{l} [8{,}37] = 8 \\ [8] = 8 \end{array} \right)$

potom $n > \frac{1}{\varepsilon}$, tedy $\frac{1}{n} < \varepsilon = 0 + \varepsilon$

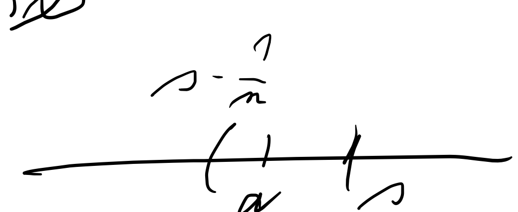
Vztah mezi limitou a supremem:

① a_n je neklesající posl. $\Rightarrow \lim\limits_{n \to \infty} a_n = \sup\{a_n, n \in \mathbb{N}\}$

(omezená)



② $\exists \{a_n\}_{n=1}^{\infty}$, $a_n \in M$, splňující $\lim\limits_{n \to \infty} a_n = \sup M$



$M_3 = \mathbb{Q} \cap [0, 1)$

$\sup M_3 = 1$: 1. $\forall x \in M_3 : x \leq 1$ zrejmé

2. nech $\varepsilon > 0$. Z hustoty

$\mathbb{Q}$ nájdeme $q \in \mathbb{Q}$ splňujúce

$q \in (1-\varepsilon, 1)$, teda $q \in M_3$

a $q > 1 - \varepsilon$.

$\inf M_3 = 0$: 1. $\forall x \in M_3 : x \geq 0$ zrejmé

2. nech $\varepsilon > 0$. Z hustoty

$\mathbb{Q}$ nájdeme $q \in \mathbb{Q}$, $q \in (0, \varepsilon)$,

tak $q \in M_3$ a $q < 0 + \varepsilon$

$M = (0, 1)$

$\sup M = 1$, ozn. $s = \sup M = 1$.

Cílem je ukázat:

1. $\forall x \in M: x \leq s$
2. $\forall \varepsilon > 0 \ \exists x \in M: x > s - \varepsilon$

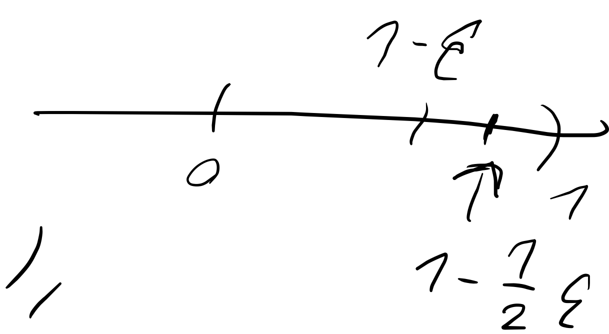
2. $(\Rightarrow)$ Kdykoli mi nepřítel zadá $\varepsilon > 0$, já umím najít $x \in M$ splňující

tj. $x > s - \varepsilon$

$(\Leftrightarrow)$ Kdykoli mi nepřítel zadá $\varepsilon > 0$, já umím najít $x \in (0,1)$ splňující

$x > 1 - \varepsilon$.

Já zadám $\varepsilon > 0$, jaké bude $x = 1 - \frac{1}{2}\varepsilon$



$\varepsilon = \frac{1}{10}$, chci $x \in (1 - \frac{1}{10}, 1) = (\frac{9}{10}, 1)$,

třeba $x = 1 - \frac{1}{20} = \frac{19}{20}$

$$M_2 = \left\{ \frac{1}{\sqrt{n}},\ n \in \mathbb{N} \right\}$$

$\inf M_2 = 0$: 1. $\forall n \in \mathbb{N}$: $\frac{1}{\sqrt{n}} \geq 0$

2. Chceme k $\varepsilon > 0$ nájsť

$x \in M_2$ splňujúce $x < 0 + \varepsilon$

Inými slovami, chceme

$n \in \mathbb{N}$, aby $\frac{1}{\sqrt{n}} < 0 + \varepsilon = \varepsilon$

Volme $n = \left(\left[\frac{1}{\varepsilon}\right] + 1\right)^2$, potom

$$\boxed{\sqrt{n} > \frac{1}{\varepsilon}} \iff \frac{1}{\sqrt{n}} < \varepsilon$$

Chci $n \in \mathbb{N}$, aby $\frac{1}{\sqrt{n}} < \varepsilon$.

Zvolíme $n = \left(\left[\frac{1}{\varepsilon}\right] + 1\right)^2$

$$\sqrt{n} = \left[\frac{1}{\varepsilon}\right] + 1 > \frac{1}{\varepsilon}$$

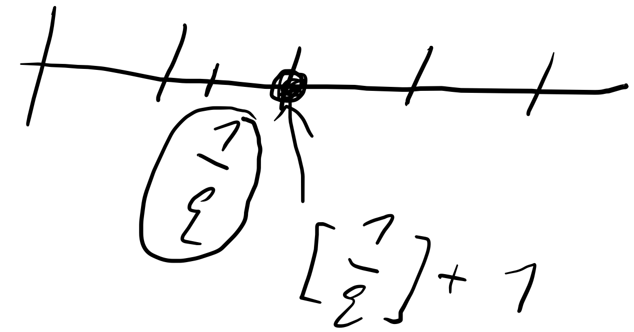
$$\Longleftrightarrow$$

$$\frac{1}{\sqrt{n}} < \varepsilon$$

Zvolíme $n = \left[\frac{1}{\varepsilon}\right]^2 + 1$, tak

$$\sqrt{n} = \sqrt{\left[\frac{1}{\varepsilon}\right]^2 + 1} > \frac{1}{\varepsilon}$$

Chcę $n \in \mathbb{N}$, aby $\frac{1}{n} < \varepsilon$,



$\iff n > \frac{1}{\varepsilon}$,

wielka $n = \left[\frac{1}{\varepsilon}\right] + 1$

$\inf \{ \frac{1}{\sqrt{n}}, n \in \mathbb{N} \} = 0$,

1. $\forall n: \frac{1}{\sqrt{n}} \geq 0$

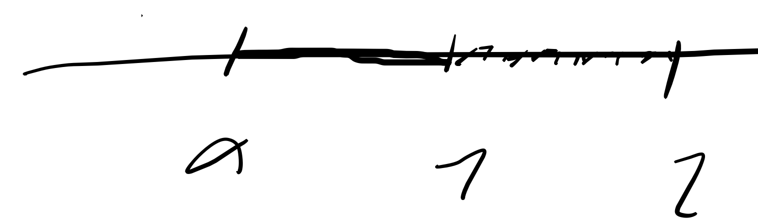
2. Chcę $n \in \mathbb{N}$, aby $\frac{1}{\sqrt{n}} > \varepsilon$

$$\sqrt{n} > \frac{1}{\varepsilon} \quad /^2$$

$$n > \frac{1}{\varepsilon^2} = \left(\frac{1}{\varepsilon}\right)^2$$

weźmy $n = \left[\left(\frac{1}{\varepsilon}\right)^2\right] + 1$

$A = (0,1) \cup (\mathbb{Q} \cap (1,2))$



$B = (0,1) \cup \{ 2 - \frac{1}{\pi n}, n \in \mathbb{N}, n > 10 \}$

inac.

$A \cap B = (0,1)$, $\sup A = \sup B = 2$

$\sup A \cap B = 1$

weźmy $\exists x \in (y,z)$, $y < z$

przykładowo weźmy

$x = \frac{y+z}{2}$

$\sup (f+g)(M) = \sup h(M),$

$(f+g)(x) = f(x) + g(x),$

tedy $(f+g)$ je zobrazení;

dejme tomu $(f+g) = h : M \to R$

$h(M) = \{ h(x),\ x \in M \}.$