CHARLES UNIVERSITY PRAGUE

faculty of mathematics and physics

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Derived equivalences induced by big tilting modules

(joint with Leonid Positselski)

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1. $\mathscr{H} = \mathscr{D}^{\leq 0} \cap \mathscr{D}^{\geq 0}$ is an abelian category (the heart of the t-str.),
2. for each $i \geq 0$ there is a natural transformations

$$\vartheta^i_{X,Y} \colon \mathrm{Ext}^i_{\mathscr{H}}(X, Y) \to \mathscr{D}(X, Y[i]),$$

3. $\vartheta^0_{X,Y}$ and $\vartheta^1_{X,Y}$ are iso's, $\vartheta^2_{X,Y}$ is mono.

Derived equivalences via realization functors

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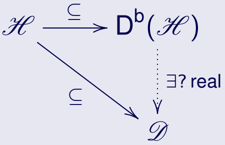

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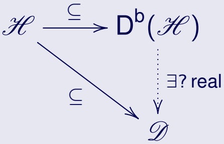

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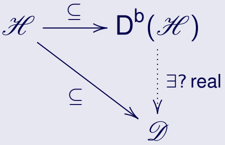

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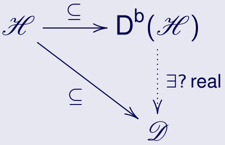

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The tilting-cotilting correspondence

Corollary

There is a bijective correspondence between

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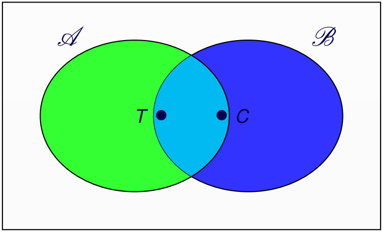

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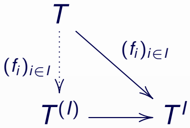

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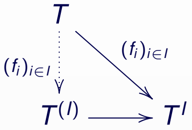

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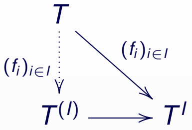

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- Note: We have a forgetful functor $\mathrm{Contra}\,S \to \mathrm{Mod}\,S$.

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The relation to results of Bazzoni-Mantese-Tonolo: ▸ to motivation

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Thank you for your attention!