Localizations of the derived category of a valuation domain

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Outline

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Valuation domains

Definition

A valuation domain is a commutative (not necessarily noetherian!) domain whose ideals are totally ordered by $\subseteq$.

Examples (trivial)

Discrete valuation domains: $\mathbb{Z}_{(p)}$ (p a prime number), $k[\![x]\!]$ (k a field).

The Goal (to be explained)

Classify all smashing localizations of the unbounded derived category $D(\mathrm{Mod}R)$ of a valuation domain R. We restrict to valuation domains with finite Zariski spectrum at the moment.

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More interesting examples

Construction

Let k be a field and $(G, +, \leq)$ a totally ordered abelian group. Denote by $G_{\geq 0} = \{g \in G \mid g \geq 0\}$ the non-negative cone and by $G_{>0}$ the subsemigroup of all positive elements.

Consider the monoid ring $S = k[G_{\geq 0}]$: The k-subspace $\mathfrak{m} = k[G_{>0}]$ is a maximal ideal of S and the localization $R = S_{\mathfrak{m}}$ is a valuation domain.

Examples

1. For $(G, +, \leq) = (\mathbb{Z}, +, \leq)$ we get $R \cong k[x]_{(x)}$ (a discrete VD).
2. For $(G, +, \leq) = (\mathbb{Q}, +, \leq)$ we get R with $\operatorname{Spec} R = \{0, \mathfrak{m}\}$, but $\mathfrak{m}^2 = \mathfrak{m}$! (the ring of Puiseux series has similar properties)
3. For $(G, +, \leq) = (\mathbb{Q}^n, +, \leq_{\text{lex}})$ we get R with
$$\operatorname{Spec} R: \quad 0 = \mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_n = \mathfrak{m} \quad \text{and} \quad (\forall j)(\mathfrak{p}_j^2 = \mathfrak{p}_j).$$
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1. Valuation domains
2. A hierarchy of triangulated localizations
3. Examples
4. About the proof

The structure on D(Mod$R$)

Fact

If R is a commutative ring, then D(Mod$R$) is a compactly generated tensor triangulated category.

- D(Mod$R$) is triangulated, the suspension functor $\Sigma\colon \mathrm{D}(\mathrm{Mod}R) \to \mathrm{D}(\mathrm{Mod}R)$ shifts complexes

$$X\colon \qquad \cdots \to X^{-1} \to X^{0} \to X^{1} \to \dots$$

  to the left and changes sings of the differentials.
- $(\mathrm{D}(\mathrm{Mod}R), \otimes_R^L, R)$ is a symmetric monoidal category, where $\otimes_R^L$ denotes the left derived functor of the tensor product. Moreover, $\otimes_R^L$ is exact in each variable.
- There is a set $\mathcal{S}$ of objects of D(Mod$R$) such that each $S \in \mathcal{S}$ is compact (that is, $\mathrm{Hom}(S,-)\colon \mathrm{D}(\mathrm{Mod}R) \to \mathrm{Ab}$ preserves coproducts) and for each $0 \neq X \in \mathrm{D}(\mathrm{Mod}R)$ there exists $0 \neq f\colon S \to X$ with $S \in \mathcal{S}$. For instance $\mathcal{S} = \{R[n] \mid n \in \mathbb{Z}\}$.

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Types of localization

- We would like to understand the structure of $\mathrm{D}(\mathrm{Mod}R)$. It is hopeless to classify objects, but we may classify kernels of various triangulated functors.
- We have

$$\left\{ \begin{array}{c} \text{Coproduct preserving Verdier localizations} \\ \mathrm{D}(\mathrm{Mod}R) \to \mathrm{D}(\mathrm{Mod}R)/\mathcal{L} \end{array} \right\}$$

$$\bigcup$$

$$\{ \text{ Bousfield localizations } L\colon \mathrm{D}(\mathrm{Mod}R) \to \mathrm{D}(\mathrm{Mod}R) \}$$

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Classifiable!

Types of localization

- We would like to understand the structure of $\mathrm{D}(\mathrm{Mod}R)$. It is hopeless to classify objects, but we may classify kernels of various triangulated functors.
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Thomason's classification of finite localizations

Theorem (Thomason, 1997)

Let R be a commutative ring. Then there is a bijection between

1. *compactly generated localizations $L\colon \mathrm{D}(\mathrm{Mod}R) \to \mathrm{D}(\mathrm{Mod}R)$;*
2. *Thomason subsets of $\mathrm{Spec}\,R$.*

Definition

A subset $U \subseteq \mathrm{Spec}\,R$ is a Thomason set if U is a union of Zariski closed sets of $\mathrm{Spec}\,R$ with quasi-compact complements.

Example

Let R be a valuation domain with $\mathrm{Spec}\,R\colon 0 = \mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_n = \mathfrak{m}$. Then the Thomason sets are simply upper sets with respect to $\subseteq$. The corresponding localization for $\{\mathfrak{p}_j, \mathfrak{p}_{j+1}, \dots, \mathfrak{p}_n\} \subseteq \mathrm{Spec}\,R$ with $j \geq 1$ is

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A functor $L\colon \mathrm{D}(\mathrm{Mod}R) \to \mathrm{D}(\mathrm{Mod}R)$ is a smashing localization if there is $C \in \mathrm{D}(\mathrm{Mod}R)$ and a morphism $\eta\colon R \to C$ such that

- $L \cong C \otimes_R^L -$, and
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Remarks

1. The term smashing comes from the stable homotopy category, where the role of $\otimes_R^L$ is taken by the smash product $\wedge$.
2. Telescope conjecture (fails in general!): Every smashing localization is compactly generated.
3. Keller (1994): If R is a valuation domain with Spec $R = \{0, \mathfrak{m}\}$ and $\mathfrak{m}^2 = \mathfrak{m}$, then the telescope conjecture fails for $\mathrm{D}(\mathrm{Mod}R)$. A counterexample is $L = R/\mathfrak{m} \otimes_R^L -$. In fact, the telescope conj. fails whenever R is a VD with a non-zero idempotent prime.

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The classification of smashing localizations

Theorem (Příhoda-Š.)

Let R be a valuation domain with finite Zariski spectrum, and let $\mathcal{P} \subseteq \operatorname{Spec} R$ be the set of idempotent prime ideals. Consider $\operatorname{Spec} R$ and $\mathcal{P}$ as topological spaces where

- *open sets of $\operatorname{Spec} R$ are upper (= Thomason) subsets,*
- *open sets of $\mathcal{P}$ are lower sets.*

Then the following holds:

1. *There is an explicitly described subspace $X \subseteq \operatorname{Spec} R \times \mathcal{P}$ and a bijection between*
   - *smashing localizations $L\colon \mathrm{D}(\mathrm{Mod}R) \to \mathrm{D}(\mathrm{Mod}R)$;*
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2. *The quotient map $\pi\colon X \to \operatorname{Spec} R$ induces the inclusion*

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The description of the space X

- Suppose $\operatorname{Spec} R = \{0 = \mathfrak{p}_0, \mathfrak{p}_1, \dots, \mathfrak{p}_n = \mathfrak{m}\}$.
- Let $\mathcal{P} = \{\mathfrak{p}_{i_0}, \dots, \mathfrak{p}_{i_s}\}$ be all idempotent ideals. Here:

$$0 = i_0 < i_1 < \cdots < i_s \quad \text{and} \quad s \geq 0.$$

- Then, if we formally put $i_{s+1} = n$, we have

$$X = \{(\mathfrak{p}_j, \mathfrak{p}_{i_\ell}) \in \operatorname{Spec} R \times \mathcal{P} \mid 0 \leq \ell \leq s \ \text{ and } \ i_\ell \leq j \leq i_{\ell+1}\}.$$

- The topology has “upper left corners” as basic open sets.

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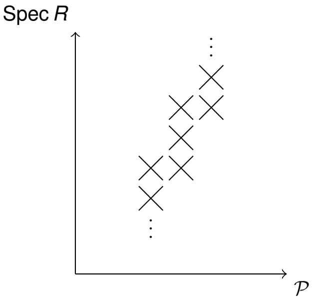

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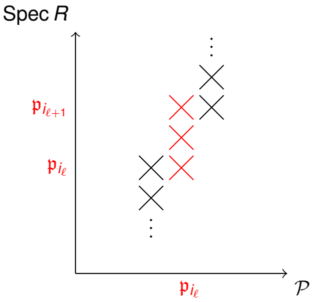

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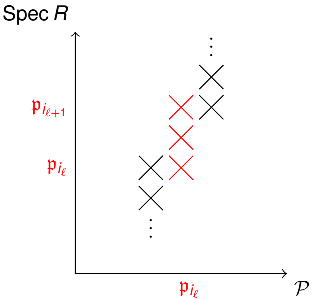

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Outline

Puiseux series

- Suppose that $\operatorname{Spec} R = \{0, \mathfrak{m}\}$ and $\mathfrak{m} = \mathfrak{m}^2$.
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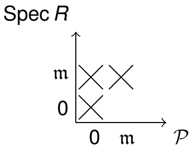

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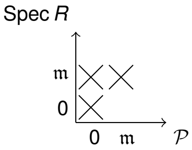

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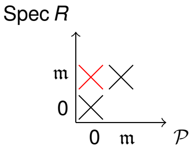

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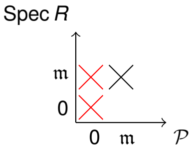

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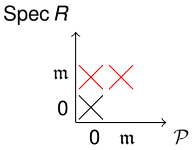

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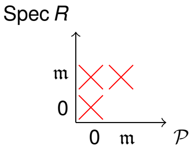

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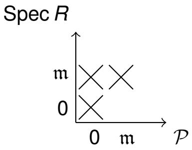

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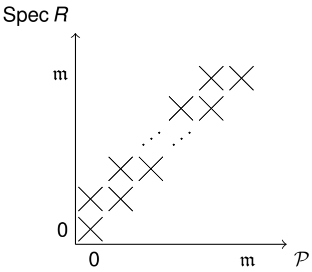

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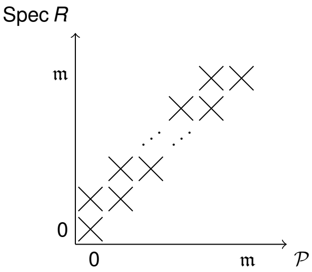

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Outline

1. Valuation domains
2. A hierarchy of triangulated localizations
3. Examples
4. About the proof

Local to global principle

- Let $\mathcal{T}$ be triangulated compactly generated (e.g. $\mathcal{T} = \mathrm{D}(\mathrm{Mod}R)$) and $L\colon \mathcal{T} \to \mathcal{T}$ be a compactly generated localization.
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Let R be a valuation domain with

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Classify all smashing localizations of

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Theorem (Krause, 2005)

Let $\mathcal{T}$ be a compactly generated triangulated category. Then there is a bijective correspondence between

1. *smashing localizations of $\mathcal{T}$ (up to natural equivalence);*
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A 2-sided ideal $\mathcal{I}$ of morphisms of $\mathcal{T}^c$ is called exact if it satisfies

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Definition

A 2-sided ideal $\mathcal{I}$ of morphisms of $\mathcal{T}^c$ is called exact if it satisfies

1. $\Sigma\mathcal{I} = \mathcal{I}$,
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The category of compact objects

- Let $\mathcal{T} = \{X \in \mathrm{D}(\mathrm{Mod}R) \mid \mathrm{Ann}(x) \supsetneq \mathfrak{p}_{n-1} \text{ for each } x \in H^*(X)\}$ as above.
- Then $\mathcal{T}^c \cong \mathrm{D}^b(\mathcal{A})$, where

$$\mathcal{A} = \{M \in \mathrm{mod}R \mid \mathrm{Ann}\, M \supsetneq \mathfrak{p}_{n-1}\}$$

  Here, mod$R$ stands for the category of all finitely presented R-modules.
- One can prove that each $M \in \mathcal{A}$ is of the form

$$M \cong \bigoplus_{i=1}^{\ell} R/(r_i) \quad \text{forsome} \quad r_i \in R \setminus \mathfrak{p}_{n-1}.$$

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Ideals in the category compact objects

Observation

There is a bijective correspondence between

- suspension invariant idempotent ideals of $\mathcal{T}^c$,
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$$\operatorname{ind}\mathcal{A} = \{R/(r) \mid r \in R \setminus \mathfrak{p}_{n-1}\} \quad (\subseteq \operatorname{mod} R)$$

Remarks

- The classification of idempotent ideals in $\operatorname{ind}\mathcal{A}$ is not straightforward, but doable. They are controlled by what we call Cauchy sequences of morphisms in $\operatorname{ind}\mathcal{A}$.
- The saturation property does not translate nicely to $\operatorname{ind}\mathcal{A}$. But keeping the correspondence above in mind, one gets that there can only be one non-trivial idempotent ideal corresponding to a saturated ideal in $\mathcal{T}^c$, and this is the case if and only if $\mathfrak{m} = \mathfrak{m}^2 (\subseteq R)$.

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