On the Finitistic Dimension Conjecture

İzmir Mathematics Days

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The finitistic dimension conjectures

A general aim

Given a ring R (associative, with unit, not necessarily commutative), we would like to understand the structure of right R-modules.

Example

- $R = k$ a field—linear algebra,
- $R = k[x]$—Jordan normal form, still linear algebra (a $k[x]$-module is the same as a vector space V with a linear operator $x \cdot - \colon V \to V$),
- $R = k[x_1, \ldots, x_n]$, $R = \left(\begin{smallmatrix} k & k \\ k & k \end{smallmatrix}\right)$, $R = \left(\begin{smallmatrix} k & 0 \\ k & k \end{smallmatrix}\right)$, $R = kG$ (G a group) ...

The projective dimension of a module

- We will try to understand a right R-module M in terms of generators and relations.
- Formally, we construct a free resolution of M:

$$\cdots \to R^{(I_2)} \to R^{(I_1)} \to R^{(I_0)} \to M \to 0$$

- Here:
  - I_0 indexes generators of M,
  - I_1 indexes relations between the generators,
  - I_2 indexes relations between the relations, ...
- Does this procedure stop?
- We define the projective dimension of M as

$$\text{proj. dim. } M = \sup\{n \mid \exists\, 0 \to P_n \to \cdots \to P_1 \to P_0 \to M \to 0\}$$

  with all P_n projective (= direct summands of free R-modules).
- If no such finite sequence exists, we put proj. dim. $M = \infty$.

The global dimension of a ring

- This allows us to define a measure of how complicated the module theory for a given ring R is:
- We define the (right) global dimension of R as

$$\operatorname{gl.dim.} R = \sup\{\operatorname{proj.dim.} M \mid M \in \text{Mod-}R\}.$$

- Fact (Baer criterion):
  $\operatorname{gl.dim.} R = \sup\{\operatorname{proj.dim.} R/I \mid I \leq R \text{ right ideal}\}$.

Theorem (Hilbert, 1890)
If k is a field, then $\operatorname{gl.dim.} k[x_1, \ldots, x_n] = n$.

An example

- Sometimes, the global dimension is not an adequate measure of complexity of Mod-R.
- Consider for example the dual numbers $R = k[x]/(x^2)$.
- Then each R-module is direct sum of copies of R and $R/(x) \cong k$, so the structure of R-modules is not much more complicated than those of vector spaces.
- On the other hand, $\operatorname{gl.dim.} R = \infty$ since $\operatorname{proj.dim.} k = \infty$:

$$\cdots \to R \xrightarrow{x} R \xrightarrow{x} R \to k \to 0$$

- Need for finer invariants!

The finitistic dimensions of a ring

Definition

1. The (right) big finitistic dimension of a ring R is defined as

$$\text{Fin.dim.}\, R = \sup\{\text{proj.dim.}\, M \mid M \in \text{Mod-}R \ \&\ \text{proj.dim.}\, M < \infty\}.$$

2. The (right) little finitistic dimension of a ring R is defined as

$$\text{fin.dim.}\, R = \sup\{\text{proj.dim.}\, M \mid M \text{ fin. gen. } \&\ \text{proj.dim.}\, M < \infty\}.$$

Warning: If R is a nice enough ring (e.g. left and right noetherian) then the left and right global dimensions are equal.

This is not true for finitistic dimensions even for finite dimensional algebras over a field!

The finitistic dimension conjectures

Conjectures (Bass, $\sim$1960)

(I) $\operatorname{fin.dim.} R = \operatorname{Fin.dim.} R$,

(II) $\operatorname{fin.dim.} R < \infty$.

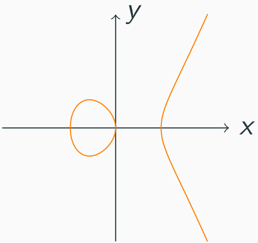
In Conjecture II, we can alternatively ask whether $\operatorname{Fin.dim.} R < \infty$.

- I will discuss the conjectures in the situations where
  1. R is a (certain) commutative noetherian ring and
  2. R is a finite dimensional algebra over a field.
- It turns out that Conjecture I is false.
- Conjecture II for finite dimensional algebras is, on the other hand, and important open problem (despite really a lot of effort to solve it)!
- Reference: B. Huisgen-Zimmermann, The finitistic dimension conjectures—a tale of 3.5 decades, in Abelian Groups and Modules, 501–517, Dordrecht (1995) Kluwer, arXiv:1407.2383.

Finitistic dimensions in commutative algebra

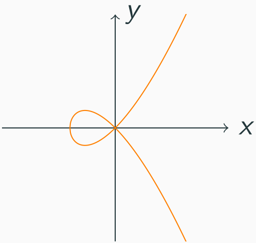
Coordinate rings of algebraic varieties

- In this part, let k be for the sake of simplicity an algebraically closed field.
- An algebraic variety $V \subseteq k^n$ is the set of zeroes of some collection of polynomials $S \subseteq k[x_1, \ldots, x_n]$.
- The real part of $V \subseteq \mathbb{C}^2$ if $k = \mathbb{C}$ may look like:



$S = \{y^2 - x(x-1)(x+1)\}$

(smooth)



$S = \{y^2 - x^2(x+1)\}$

(singular)

Coordinate rings of algebraic varieties

- Algebraic varieties $V \subseteq k^n$ always come with a coordinate ring, which is the ring of functions $\varphi\colon V \to k$ given by a polynomial in the coordinates of points of V.
- Standard fact: $k[V] \cong k[x_1, \ldots, x_n]/I(V)$, where

$$I(V) = \{f \in k[x_1, \ldots, x_n] \mid f \text{ vanishes on } V\}$$

- One can show that, in our previous examples (with $k = \mathbb{C}$), we have $\mathbb{C}[V] = \mathbb{C}[x, y]/(y^2 - x(x-1)(x+1))$ and $\mathbb{C}[V] = \mathbb{C}[x, y]/(y^2 - x^2(x+1))$, respectively.
- These rings are clearly commutative and they are noetherian by Hilbert's Basis Theorem.

The Krull dimension

In order to explain the relevance of finitistic dimensions in for coordinated rings, we need a geometric notion of dimension of a variety which works over any algebraically closed field.

Definition
Let $V \subseteq k^n$ be an algebraic variety. We say that V is irreducible if it is non-empty and we cannot write $V = V_1 \cup V_2$ with algebraic varieties $V_1, V_2 \subsetneq V$.

Definition
Let $V \subseteq k^n$ be an irreducible algebraic variety. We define the Krull dimension of V inductively as follows:

- Irreducible varieties of dimension 0 are precisely points.
- V has dimension $n > 0$ if all irreducible varieties $W \subsetneq V$ have dimension $\leq n - 1$, but V itself does not have dimension $\leq n - 1$.

The big finitistic dimension for commutative notherian rings

Theorem (Bass, 1962 and Gruson-Raynaud, 1971)
If V is an algebraic variety, then $\text{Fin. dim. } k[V]$ *equals the Krull dimension of V.*

Theorem (Auslander-Buchsbaum-Serre, 1956)
If V is an algebraic variety, then $\text{gl. dim. } k[V] < \infty$ *iff V is smooth. In that case,* $\text{gl. dim. } k[V]$ *equals the Krull dimension of V.*

Remarks

1. One can define the Krull dimension for any commutative noetherian ring R in terms of lengths of chains of prime ideals.
2. In that case, one still has that $\text{Fin. dim. } R$ equals the Krull dimension of R.
3. At this level of generality, Conjecture II fails. Nagata (1962) constructed commutative noetherian rings of infinite Krull dimension.

The little finitistic dimension

- The little finitistic dimension also has an interpretation:
- If R is commutative local noetherian, then $\text{fin.dim.}\, R$ equals the depth of R (a certain algebraic invariant of R).
- This, together with the interpretation of $\text{Fin.dim.}\, R$, shows that Conjecture I (i.e. $\text{fin.dim.}\, R = \text{Fin.dim.}\, R$), often fails for commutative noetherian local rings.
- More in detail, Conjecture I holds for R iff the Krull dimension of R equals the depth of R, which by definition means that R is a so-called Cohen-Macaulay ring.

Finite dimensional algebras

Homological conjectures

- There exist a collection of homological conjectures in representation theory of finite dimensional algebras which postulate certain homological finiteness and symmetries.
- Variants of Conjecture II are one of the strongest among these (from now on, R is possibly non-commutative finite dimensional algebra over a field k):

$$\operatorname{Fin.dim.} R < \infty \text{ for all } R$$
$$\Downarrow$$
$$\operatorname{fin.dim.} R < \infty \text{ for all } R$$
$$\Downarrow$$

Nakayama conjecture, Generalized Nakayama Conjecture,
Nunke Condition, Gorenstein Symmetry Conjecture, ...

- On the other hand, Conjecture I fails in general (Huisgen-Zimmerman, 1992; Smalø, 1998).

Known cases where $\operatorname{fin.dim.} < \infty$ holds

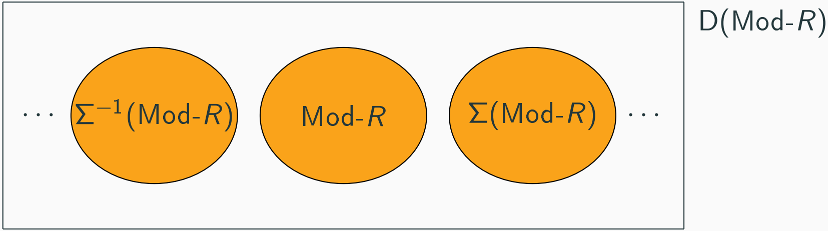
- The finiteness of the little (and sometimes also the big) finitistic dimension has been verified in various special cases:
    - If R has Loewy length ≤ 3,
    - If R is a monomial path algebra, i.e. $R = kQ/(\text{a collection of paths})$, where Q is a finite quiver,
    - ...
- There are also situations where $\operatorname{Fin.dim.} R < \infty$ for almost trivial reasons, e.g. if R is a local algebra or an Iwanaga-Gorenstein algebra.
- A plethora of reduction techniques have been developed, which roughly say that if $\operatorname{fin.dim.} R < \infty$ for an algebra R, then also $\operatorname{fin.dim.} S < \infty$ for another algebra S related to R.

The derived categories, part 1

- Given a short exact sequence $0 \to K \to L \to M \to 0$ in Mod-R and $X \in$ Mod-R, we only obtain a left exact sequence of abelian groups
$$0 \to \mathrm{Hom}_R(X, K) \to \mathrm{Hom}_R(X, L) \to \mathrm{Hom}_R(X, M).$$
- It is well known that one can complete this naturally to an infinite long exact sequence
$$\begin{aligned} 0 \to \mathrm{Hom}_R(X, K) \to \mathrm{Hom}_R(X, L) \to \mathrm{Hom}_R(X, M) \to \\ \mathrm{Ext}^1_R(X, K) \to \mathrm{Ext}^1_R(X, L) \to \mathrm{Ext}^1_R(X, M) \to \\ \mathrm{Ext}^2_R(X, K) \to \mathrm{Ext}^2_R(X, L) \to \mathrm{Ext}^2_R(X, M) \to \cdots \end{aligned}$$
- Grothendieck invented in 1958 a category D(Mod-R), which contains Mod-R and comes equipped with an autoequivalence $\Sigma\colon \mathrm{D}(\text{Mod-}R) \to \mathrm{D}(\text{Mod-}R)$ such that $\mathrm{Ext}^n_R(X, M) \cong \mathrm{Hom}_{\mathrm{D}(\text{Mod-}R)}(X, \Sigma^n M)$.

The derived categories, part 2

- An illustration:



- One constructs $\mathsf{D}(\text{Mod-}R)$ as the category of cochain complexes of modules

$$\cdots \longrightarrow X^{-1} \xrightarrow{\partial^{-1}} X^0 \xrightarrow{\partial^0} X^1 \xrightarrow{\partial^1} X^2 \longrightarrow \cdots$$

localized at all the cohomology isomorphisms.

- One can also construct a bounded version, $\mathsf{D}^{\mathsf{b}}(\text{mod-}R)$, considering only bounded complexes of finitely generated modules.

Invariance under derived equivalence

Definition (Happel, 1987; Rickard, 1989; Keller, 1994)

Algebras R and S are derived equivalent if $\mathrm{D}(\text{Mod-}R) \simeq \mathrm{D}(\text{Mod-}S)$ (this happens iff $\mathrm{D}^{\mathrm{b}}(\text{mod-}R) \simeq \mathrm{D}^{\mathrm{b}}(\text{mod-}S)$).

Theorem (Happel, 1993)

Suppose that R and S are finite dimensional algebras which are derived equivalent. Then $\operatorname{fin.dim.} R < \infty$ *iff* $\operatorname{fin.dim.} S < \infty$.

Recent developments

Rickard's injective generation, part 1

- A new member in the family of homological conjectures appeared recently (Rickard, 2019, published in Adv. Math., arXiv:1804.09801):

The injective cogenerator $E := \mathrm{Hom}_k(R, k)$ generates $\mathrm{D}(\mathrm{Mod}\text{-}R)$.

$$\Downarrow$$

$$\mathrm{Fin.dim.}\, R < \infty \text{ for all } R$$

$$\Downarrow$$

$$\mathrm{fin.dim.}\, R < \infty \text{ for all } R$$

$$\Downarrow$$

Other homological conjectures ...

- This property is actually not so difficult to check in particular examples of finite dimensional algebras!

Rickard's injective generation, part 2

- Injective generation of D(Mod-R) (Rickard, 2019, published in Adv. Math., arXiv:1804.09801):
- Put $E := \mathrm{Hom}_k(R, k) \in$ Mod-R. This is an injective cogenerator.
- We ask whether a complex $X \in$ D(Mod-R) such that $\mathrm{Hom}_{\mathrm{D(Mod}\text{-}R)}(E, \Sigma^n(X)) = 0$ for all $n \in \mathbb{Z}$ must be necessarily isomorphic to a zero complex in D(Mod-R).
- Remark: A complex $X \in$ D(Mod-R) such that $\mathrm{Hom}_{\mathrm{D(Mod}\text{-}R)}(R, \Sigma^n(X)) = 0$ for all $n \in \mathbb{Z}$, is isomorphic to a zero complex in D(Mod-R) (a basic property of D(Mod-R)).

The stable module category

- Another approach to finitistic dimension conjectures through the stable module category and singularity category:
- For each $M \in \text{mod-}R$, we can take a syzygy module $\Omega(M)$ (the module of relations):

$$0 \to \Omega(M) \to P \to M \to 0.$$

- This is not a functor $\text{mod-}R \to \text{mod-}R$:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Omega(M) & \longrightarrow & P & \longrightarrow & M & \longrightarrow & 0 \\ & & {\scriptstyle \exists^{>1}\Omega(f)}\big\downarrow & & {\scriptstyle \exists^{>1}}\big\downarrow & & \big\downarrow{\scriptstyle f} & & \\ 0 & \longrightarrow & \Omega(M) & \longrightarrow & P' & \longrightarrow & M' & \longrightarrow & 0 \end{array}$$

- But it induces a functor $\Omega\colon \underline{\text{mod-}}R \to \underline{\text{mod-}}R$, where $\underline{\text{mod-}}R$ is the factor of $\text{mod-}R$ modulo the maps which factor through a projective module.

The singularity category

- There is another inconvenience: the functor $\Omega\colon \underline{\text{mod-}}R \to \underline{\text{mod-}}R$ is usually far from being an equivalence.
- Analogy: In algebraic topology, the suspension functor $\Sigma\colon \mathrm{Top}_* \to \mathrm{Top}_*$ is not an equivalence, although one often wants it to be!
- Same solution in both cases: the stabilization construction (by Spanier-Whitehead in algebraic topology, in our situation due to Keller-Vossieck, 1987 and Beligiannis, 2000).
- The stable version of $\underline{\text{mod-}}R$ coincides with the so-called singularity category

$$\mathrm{D_{sg}}(R) := \mathrm{D^b}(\text{mod-}R)/(\text{perfect complexes}).$$

- The term "singularity category" goes back to Orlov, 2003 as $\mathrm{D_{sg}}(R) = 0$ iff $\mathrm{gl.\,dim.}\, R < \infty$, but it was studied already by Buchweitz in 1986.

Finitistic dimension as a property of D_{sg}

Theorem (Š.)
Let R be a finite dimensional algebra over a field.

The fact whether $\mathrm{Fin.dim.}\, R < \infty$ *or not is a property of* $\mathrm{D}_{\mathrm{sg}}(R^{\mathrm{op}})$ *(as a triangulated category).*

- If $\mathrm{Fin.dim.}\, R < \infty$ and $\mathrm{D}_{\mathrm{sg}}(R) \simeq \mathrm{D}_{\mathrm{sg}}(S)$ (as triangulated categories), then also $\mathrm{Fin.dim.}\, S < \infty$.
- A derived equivalence implies $\mathrm{D}_{\mathrm{sg}}(R) \simeq \mathrm{D}_{\mathrm{sg}}(S)$, so this generalizes Happel's result.
- For special equivalences $\mathrm{D}_{\mathrm{sg}}(R) \simeq \mathrm{D}_{\mathrm{sg}}(S)$, the latter was proved by Wang, 2015.
- More importantly, people started to study $\mathrm{D}_{\mathrm{sg}}(R)$ recently—see Chen-Wang, 2021, arXiv:2109.11278!