Tilting modules—homological algebra and structure

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Outline

1 Modules
   • Definition and interpretation
   • Homological algebra

2 Tilting modules and tilting classes
   • Introduction
   • Finite type
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   • Definition and interpretation
   • Homological algebra

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   • Introduction
   • Finite type
What is a module?

Given a (non-commutative) ring \((\mathbb{R}, +, -, 0, \cdot, 1)\), we define a module as a "vector space" over \(\mathbb{R}\):

**Definition**

A left \(\mathbb{R}\)-module is an abelian group \((M, +, -, 0)\) such that we can multiply each \(r \in \mathbb{R}\) and \(m \in M\) and the following axioms hold:

1. \(r(m + m') = rm + rm'\),
2. \((r + s)m = rm + sm\),
3. \(r(sm) = (rs)m\),
4. \(1 \cdot m = m\).

Module are far reaching generalization of vector spaces, their properties heavily depend on the ring \(\mathbb{R}\)!
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A ring is a left module over itself, \( 1 \in R \) forms a free basis.

Free modules are modules with a free basis, they are isomorphic to \( R^{(I)} \), where \( I \) is a set.

A module \( P \) is projective if it is a direct summand of a free module, that is, \( P \oplus Q = R^{(I)} \) for some \( Q \) and \( I \).

Each module is of the form \( F/K \), where \( F \) is a free module.
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Modules over specific rings—vector bundles

Take a compact Hausdorff space:

\[ M : \]

Let \( R \) be the ring of all continuous real (or complex) valued functions on \( M \).

**Theorem (Swan, 1961)**

There is a bijective correspondence between isomorphism classes of vector bundles on \( M \) and isomorphism classes of finitely generated projective \( R \)-modules.
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There is a bijective correspondence between isomorphism classes of vector bundles on $M$ and isomorphism classes of finitely generated projective $R$-modules.
Let $k$ be a field and $Q$ be a quiver:

\[ \bullet\xrightarrow{\alpha_2} \bullet \xleftarrow{\alpha_1} \bullet \xrightarrow{\alpha_3} \bullet \xleftarrow{\alpha_4} \bullet \]

Let $R = kQ$ be the path algebra, that is, a ring defined by the “$k$-linear extension of compositions of paths”. Then left $R$-modules correspond to diagrams of the form:

\[ U \xleftarrow{f_2} V \xrightarrow{f_1} W \]

\[ U \xleftarrow{f_3} W \xrightarrow{f_4} V \]

Classification of modules corresponds to finding normal forms for matrices, eg. pairs of bilinear forms, 4-subspace problem.
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If $R$ is a ring and $M, N$ are $R$-modules, denote by $\text{Hom}_R(M, N)$ the set of all homomorphisms from $M$ to $N$. This is an abelian group via $(f + g)(m) = f(m) + g(m)$.

Another important concept: extensions. Let $\text{Ext}_R^1(M, N)$ be the set of all short exact sequences

$$
\varepsilon : 0 \longrightarrow N \overset{i}{\longrightarrow} E \overset{p}{\longrightarrow} M \longrightarrow 0
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modulo the relation identifying $\varepsilon$ and $\varepsilon'$ when

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This is an abelian group via the so called Baer sum.
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There is a close relation between Hom and Ext which leads to the definition of higher Ext groups.

For modules $M, N$ and $n \geq 1$ we define $\text{Ext}_R^n(M, N)$ to be the set

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modulo a certain equivalence relation.

$\text{Ext}_R^n(M, N)$ is again naturally an abelian group.

Projective modules are important in computing these Ext-groups.

Fact:

$P$ is a projective module $\iff$

$\text{Ext}_R^1(P, M) = 0$ for each module $M$ $\iff$

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Generalization of projective modules

Definition

Let \( R \) be a ring. A module \( T \) is called \textit{tilting} if

1. \( \text{Ext}^n(T^{(I)}, T^{(I)}) = 0 \) for each set \( I \),
2. there exists an exact sequence of the form
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Remarks:

- Finitely generated tilting give connection between modules over \( R \) and \( S = \text{End}_R(T) \) (via so called derived equivalences).
- Infinitely generated tilting modules give insight into the structure of infinitely generated modules and into behavior of Ext-functors.
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3. there exists an exact sequence of the form
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Remarks:

- Finitely generated tilting give connection between modules over $R$ and $S = \text{End}_R(T)$ (via so called derived equivalences).
- Infinitely generated tilting modules give insight into the structure of infinitely generated modules and into behavior of Ext-functors.
Generalization of projective modules

Definition

Let \( R \) be a ring. A module \( T \) is called \textit{tilting} if

1. \( \text{Ext}^n(T(I), T(I)) = 0 \) for each set \( I \),
2. there exists an exact sequence of the form
\[
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### Definition

Let $R$ be a ring and $T$ a tilting module. Then

$$\mathcal{T} = \{ M \mid \text{Ext}^n(T, M) = 0 \text{ for each } n \geq 1 \}$$

is called the tilting class corresponding to $T$.

### Remarks:

- Tilting classes are often easier to deal with. Often we know we have a tilting class, but do not understand the tilting module.
- We have for each $P \in \mathcal{T}$:
  
  $P$ is a summand in some $T^{(l)}$ $\iff$
  
  $\text{Ext}^1_R(P, M) = 0$ for each module $M \in \mathcal{T}$ $\iff$
  
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Tilting classes

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Theorem (Bazzoni, Eklof, Š., Trlifaj, 2005)

Let $R$ be a ring and $T$ a tilting class of $R$ modules. Then $T$ is of finite type. That is, there is a set $S$ of strongly finitely presented modules such that

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That is, we have replaced

1. $\text{Ext}^n_R$ for $n \geq 1$ by just $\text{Ext}^1_R$ and
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Characterization of tilting classes over a given ring $R$.

Connection to model theory: Each tilting class $\mathcal{T}$ is axiomatizable in the language $\mathcal{L}_R$ of left $R$-modules (with function symbols $+$, $-$, 0 and $r \cdot -$ for each $r \in R$). This is because for a fixed strongly finitely presented module $S$, the condition $\text{Ext}^1(S, M) = 0$ for given $M$ can be expressed by a first order formula in $\mathcal{L}_R$.

Structure of tilting modules: Given a tilting class $\mathcal{T} = \{ M \mid \text{Ext}^1_R(S, M) = 0 \text{ for each } S \in S \}$, there is a tilting module $T$ for $\mathcal{T}$, which is the union of a transfinite smooth chain

$$0 = T_0 \subseteq T_1 \subseteq T_2 \subseteq \cdots \subseteq T_\alpha \subseteq T_{\alpha+1} \subseteq \cdots \subseteq T_\sigma = T$$

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Consequences

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