Tilting modules—homological algebra and structure

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Outline

1. Modules
    - Definition and interpretation
    - Homological algebra

2. Tilting modules and tilting classes
    - Introduction
    - Finite type

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   - Finite type

What is a module?

Given a (non-commutative) ring $(R, +, -, 0, \cdot, 1)$, we define a module as a "vector space" over R:

Definition

A left R-module is an abelian group $(M, +, -, 0)$ such that we can multiply each $r \in R$ and $m \in M$ and the following axioms hold:

1. $r(m + m') = rm + rm'$,
2. $(r + s)m = rm + sm$,
3. $r(sm) = (rs)m$,
4. $1 \cdot m = m$.

Module are far reaching generalization of vector spaces, their properties heavily depend on the ring R!

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Free and projective modules

- A ring is a left module over itself, $1 \in R$ forms a free basis.
- Free modules are modules with a free basis, they are isomorphic to $R^{(I)}$, where I is a set.
- A module P is projective if it is a direct summand of a free module, that is, $P \oplus Q = R^{(I)}$ for some Q and I.
- Each module is of the form F/K, where F is a free module.

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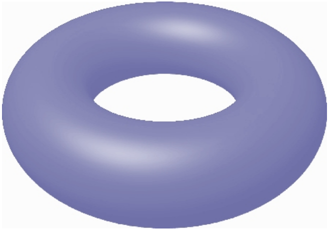
Modules over specific rings—vector bundles

Take a compact Hausdorff space:

$\mathcal{M}$:



Let R be the ring of all continuous real (or complex) valued functions on $\mathcal{M}$.

Theorem (Swan, 1961)

There is a bijective correspondence between isomorphism classes of vector bundles on $\mathcal{M}$ and isomorphism classes of finitely generated projective R-modules.

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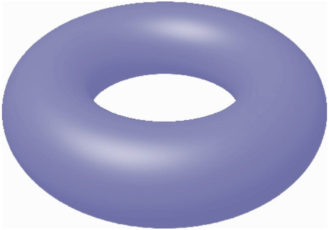

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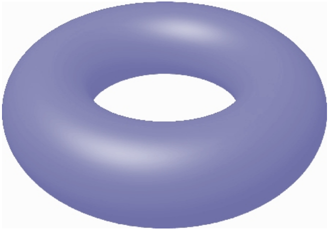

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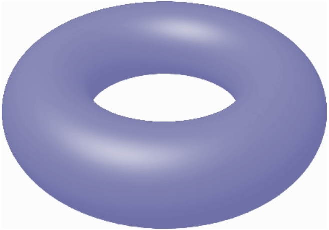

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Modules over specific rings—matrix problems

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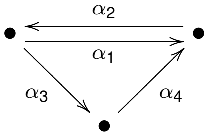

Let $R = kQ$ be the path algebra, that is, a ring defined by the "k-linear extension of compositions of paths". Then left R-modules correspond to diagrams of the form:

Classification of modules corresponds to finding normal forms for matrices, eg. pairs of bilinear forms, 4-subspace problem.

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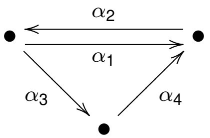

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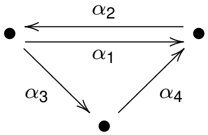

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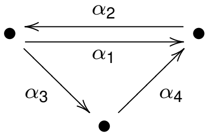

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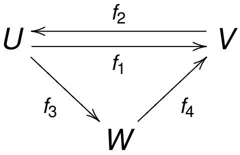

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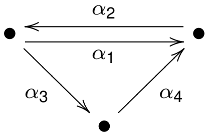

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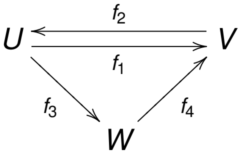

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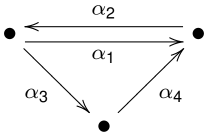

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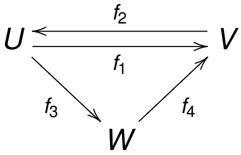

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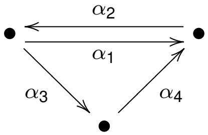

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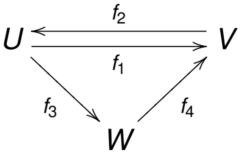

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Hom and Ext

- If R is a ring and M, N are R-modules, denote by $\mathrm{Hom}_R(M, N)$ the set of all homomorphisms from M to N. This is an abelian group via $(f + g)(m) = f(m) + g(m)$.
- Another important concept: extensions. Let $\mathrm{Ext}^1_R(M, N)$ be the set of all short exact sequences

$$\varepsilon: \quad 0 \longrightarrow N \xrightarrow{\ i\ } E \xrightarrow{\ p\ } M \longrightarrow 0$$

modulo the relation identifying ε and ε' when

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This is an abelian group via the so called Baer sum.

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Hom and Ext continued

- There is a close relation between Hom and Ext which leads to definition of higher Ext groups.
- For modules M, N and $n \geq 1$ we define $\mathrm{Ext}^n_R(M, N)$ to be the set

$$0 \longrightarrow N \longrightarrow E_1 \longrightarrow E_2 \longrightarrow \dots \longrightarrow E_n \longrightarrow M \longrightarrow 0$$

  modulo a certain equivalence relation.
- $\mathrm{Ext}^n_R(M, N)$ is again naturally an abelian group.
- Projective modules are important in computing these Ext-groups.
- Fact:

$$P \text{ is a projective module} \iff$$
$$\mathrm{Ext}^1_R(P, M) = 0 \text{ for each module } M \iff$$
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Generalization of projective modules

Definition

Let R be a ring. A module T is called tilting if

1. $\mathrm{Ext}^n(T^{(I)}, T^{(I)}) = 0$ for each set I,
2. there exists an exact sequence of the form
$$0 \longrightarrow R^{(J_1)} \longrightarrow R^{(J_2)} \longrightarrow \dots \longrightarrow R^{(J_n)} \longrightarrow T \longrightarrow 0,$$
3. there exists an exact sequence of the form
$$0 \longrightarrow R \longrightarrow T^{(K_1)} \longrightarrow T^{(K_2)} \longrightarrow \dots \longrightarrow T^{(K_m)} \longrightarrow 0.$$

Remarks:

- Finitely generated tilting give connection between modules over R and $S = \mathrm{End}_R(T)$ (via so called derived equivalences).
- Infinitely generated tilting modules give insight into the structure of infinitely generated modules and into behavior of Ext-functors.

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Tilting classes

Definition

Let R be a ring and T a tilting module. Then

$$\mathcal{T} = \{M \mid \mathrm{Ext}^n(T, M) = 0 \text{ for each } n \geq 1\}$$

is called the tilting class corresponding to T.

Remarks:

- Tilting classes are often easier to deal with. Often we know we have a tilting class, but do not understand the tilting module.
- We have for each $P \in \mathcal{T}$:

$$P \text{ is a summand in some } T^{(I)} \iff$$

$$\mathrm{Ext}^1_R(P, M) = 0 \text{ for each module } M \in \mathcal{T} \iff$$

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Finite type

Theorem (Bazzoni, Eklof, Š., Trlifaj, 2005)

Let R be a ring and $\mathcal{T}$ a tilting class of R modules. Then $\mathcal{T}$ is of finite type. That is, there is a set $\mathcal{S}$ of strongly finitely presented modules such that

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That is, we have replaced

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Consequences

- Characterization of tilting classes over a given ring R.
- Connection to model theory: Each tilting class $\mathcal{T}$ is axiomatizable in the language $\mathcal{L}_R$ of left R-modules (with function symbols $+$, $-$, 0 and $r \cdot -$ for each $r \in R$).
  This is because for a fixed strongly finitely presented module S, the condition $\mathrm{Ext}^1(S, M) = 0$ for given M can be expressed by a first order formula in $\mathcal{L}_R$.
- Structure of tilting modules: Given a tilting class $\mathcal{T} = \{M \mid \mathrm{Ext}^1_R(S, M) = 0 \text{ for each } S \in \mathcal{S}\}$, there is a tilting module T for $\mathcal{T}$, which is the union of a transfinite smooth chain

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