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Flat epimorphisms of commutative noetherian rings

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Universal localizations—general facts

- Classical localization: start with a (non-commutative!) ring A and try to make a set of elements universally invertible.
- Bergman, Cohn, Schofield: make a set of matrices of over A invertible (matrices = right A-module maps $A^n \to A^m$).
- More generally, make a set of maps $\sigma\colon P \to Q$ between finitely generated projective left A-modules invertible. Such ring morphisms are called universal localizations and always exist:

Theorem (Schofield, 1985)

Let Σ be a set of morphisms between finitely generated projective A-modules. Then there exists a ring homomorphism $f\colon A \to A_\Sigma$ such that

1. *$\sigma \otimes_A A_\Sigma$ is invertible for each $\sigma \in \Sigma$,*
2. *f is an initial ring homomorphism with this property.*

Moreover, f is epimorphism in the category of rings and $\mathrm{Tor}_1^A(A_\Sigma, A_\Sigma) = 0$.

Universal localizations—flatness

- Specific to commutative rings A:
  1. An $n \times n$ matrix M is invertible if and only if $\det M$ is invertible. So inverting square matrices is the same as inverting elements.
  2. A non-square matrix cannot be invertible unless $A = 0$ (commutative rings have invariant basis number).
- Consequence: If $A \to A_\Sigma$ is a universal localization and $\mathfrak{p} \in \operatorname{Spec} A$, then $A_\mathfrak{p} \to (A_\Sigma)_\mathfrak{p} \simeq (A_\mathfrak{p})_{\Sigma_\mathfrak{p}}$ is a classical localization, so flat over A.

Theorem (AH-M-Š-T-V)

If $A \to A_\Sigma$ is a universal localisation with A commutative, then A_Σ is also commutative and flat over A.

- For commutative noetherian rings, we have even more:

Theorem (AH-M-Š-T-V)

If A is commutative noetherian and $A \to B$ is a ring epimorphism with $\operatorname{Tor}_1^A(B, B) = 0$, then B is commutative and flat over A.

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Two inclusions

- For a commutative ring we have inclusions

$$\left\{\begin{matrix}\text{equiv. classes}\\ \text{of classical}\\ \text{localisations}\\ f\colon A \longrightarrow B\end{matrix}\right\} \subseteq \left\{\begin{matrix}\text{equiv. classes}\\ \text{of universal}\\ \text{localisations}\\ f\colon A \longrightarrow B\end{matrix}\right\} \subseteq \left\{\begin{matrix}\text{equiv. classes}\\ \text{of flat}\\ \text{epimorphisms}\\ f\colon A \longrightarrow B\end{matrix}\right\}$$

- The aim is to understand when are these inclusions equalities.
- It turns out that the answer is controlled by groups of divisors of A.

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Reduction to projectives of rank one

- Assume that $\operatorname{Spec} A$ is connected (if A is commutative noetherian, then it is a finite product of rings with connected spectra).
- Then each finitely generated projective A-module P has a rank, i.e. there is $n \geq 0$ such that $P_\mathfrak{p} \cong A_\mathfrak{p}^n$ for each $\mathfrak{p} \in \operatorname{Spec} A$.

Lemma

Let $\sigma \colon P \to Q$ be a map between finitely generated projectives.

- *If $\operatorname{rk} P = \operatorname{rk} Q$, then σ is invertible if and only if $\wedge^n \sigma$ is invertible (and $\wedge^n \sigma$ is a map between projectives of rank one).*
- *If $\operatorname{rk} P = \operatorname{rk} Q$, then σ is not invertible unless $A = 0$.*

- Upshot: Any universal localization of a commutative ring with connected spectrum is w.l.o.g. given by maps between rank one projective modules.

The Picard group

- Rank one projectives are also called invertible modules (because $P \otimes_A P^* \simeq A$, where $P^* = \mathrm{Hom}_A(P, A)$) or line bundles (geometric interpretation).
- Isoclasses of invertible modules with $\otimes$ form the so-called Picard group $\mathrm{Pic}(A)$ of A. (If A is a Dedekind domain, this is the usual ideal class group.)

Theorem (AH-M-Š-T-V)

Let A be commutative noetherian.

1. *Universal localizations are all classical if $\mathrm{Pic}(A)$ is torsion.*
2. *For locally factorial (e.g. regular) rings, the converse also holds.*

- Remark 1: Any abelian group can be $\mathrm{Pic}(A)$ for a Dedekind domain [Claborn].
- Remark 2: Universal localizations of commutative noetherian rings universally invert sections of line bundles over $\mathrm{Spec}\, A$ rather than only elements (= sections of the structure sheaf).

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Height one

Theorem (Gabriel, 1962 and Lazard, 1969)

Let A be commutative noetherian and $f\colon A \to B$ a flat ring epimorphism.

1. *$f^\flat\colon \operatorname{Spec} B \to \operatorname{Spec} A$ induces a homeomorphism onto its image (as a subspace of $\operatorname{Spec} A$).*
2. *The subset $V = \operatorname{Spec} A \setminus \operatorname{im} f^\flat$ is closed under specialization and determines f up to equivalence.*

- The following theorem extends observations by Krause and Iyengar, the proof uses properties of local cohomology:

Theorem (AH-M-Š-T-V)

Let $f\colon A \to B$ be a flat ring epimorphism with A commutative noetherian and let $V = \operatorname{Spec} A \setminus \operatorname{im} f^\flat$. Then the minimal (=most generic) primes in V are of height at most one.

The one-dimensional and locally factorial cases

Theorem (AH-M-Š-T-V)

Let A be a commutative noetherian ring. If A is one-dimensional or locally factorial, then every flat epimorphism $A \to B$ is a universal localization.

Sketch proof.

Let $f \colon A \to B$ be a flat ring epimorphism and let $V = \operatorname{Spec} A \setminus \operatorname{im} f^\flat$.

If A is locally factorial and $\mathfrak{p} \in \operatorname{Spec} A$ has height at most 1, then $\mathfrak{p}$ is projective over A. In this case, we universally localize at

$$\Sigma = \{\mathfrak{p} \overset{\subseteq}{\to} A \mid \mathfrak{p} \in V \text{ minimal}\}.$$

The one-dimensional case is more technical, it uses prime avoidance. □

The divisor class group

- Let A be a commutative noetherian ring and K its classical ring of fractions (localize at the non-zero divisors of A).
- There is another group of divisors, Weil divisors. We let $\mathrm{Div}(A)$ be the free abelian group with height one primes of A as a basis.
- Principal divisors: If $x \in K^\times$, then $\mathrm{div}(x) \in \mathrm{Div}(A)$ counts the multiplicities zeros and poles of x along height one primes.
- The divisor class groups: $\mathrm{Cl}(A) = \mathrm{Div}(A)/\{\mathrm{div}(x) \mid x \in K^\times\}$.

Facts

There is a canonical group homomorphism $\mathrm{div}\colon \mathrm{Pic}(A) \to \mathrm{Cl}(A)$.

1. If A is locally factorial (e.g. Dedekind domain), then div is an isomorphism.
2. If A is normal (= a finite product of integrally closed domains), then div is injective.

Flat ring epimorphism for normal rings

Theorem (AH-M-Š-T-V)

Let A be a commutative noetherian normal ring such that $\mathrm{Cl}(A)/\mathrm{Pic}(A)$ is torsion (e.g. if A is locally factorial). Then every flat epimorphism $A \to B$ is a universal localization.

- Remark 1: The converse holds for two-dimensional normal finitely generated algebras over a field (and in some other situations).
- Remark 2: We have various examples illustrating the necessity of most of our assumptions.

Thank you for your attention!