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Noncommutative algebraic geometry based on quantum flag manifolds

Part III.

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Coherent sheaves after Pali

- Recall: If V is a compact complex manifold, then a holomorphic vector bundle $p\colon E \to V$ can be equivalently given via a flat connection

$$\nabla_E\colon \Gamma^\infty(E) \to \Gamma^\infty(E) \otimes_{C^\infty(V)} \Omega^{(0,1)}.$$

(Koszul and Malgrange, 1958).
- There is a generalization for coherent sheaves over V:

Theorem (Pali, 2006)

Given a compact complex manifold V and the sheaf $\mathcal{O}_V^\infty$ of smooth complex-valued functions, there is a bijective correspondence between

1. *analytic coherent sheaves on V and*
2. *flat connections $\nabla_{\mathscr{G}}\colon \mathscr{G} \to \mathscr{G} \otimes_{\mathcal{O}_V^\infty} \Omega^{(0,1)}$, where the sheaf $\mathcal{O}_V^\infty$-modules locally admits a resolution*

$$0 \to (\mathcal{O}_V^\infty|_U)^{n_k} \to \cdots \to (\mathcal{O}_V^\infty|_U)^{n_1} \to (\mathcal{O}_V^\infty|_U)^{n_0} \to \mathscr{G}|_U \to 0.$$

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- If V embeds into a projective space, then V is a smooth complex projective algebraic variety (Chow, 1949).
- In that case, the categories of analytic and algebraic coherent sheaves are equivalent (Serre's GAGA Theorem, 1956).
- Pali actually proves that $\operatorname{coh} V$ is equivalent to the category of the flat connections as above.

The category of Beggs and Smith

- This motivated Beggs and Smith (2012) to define an abelian category category $\mathrm{Hol}(A)$ for a non-commutative complex structure $(\Omega^\bullet(A), \mathrm{d} = \partial + \overline{\partial})$ (e.g. $A = \mathcal{O}_q(\mathrm{Gr}_{n,r})$ as before):
- The objects are flat connections $\nabla_M \colon M \to M \otimes_A \Omega^{(0,1)}$ and the morphisms are given by $f \colon M \to N$ such that

$$\begin{array}{ccc} M & \xrightarrow{\nabla_M} & M \otimes_A \Omega^{(0,1)} \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f \otimes \Omega^{(0,1)}} \\ N & \xrightarrow[\nabla_N]{} & N \otimes_A \Omega^{(0,1)}. \end{array}$$

- First approximation to the differential description of the category of coherent sheaves: require that M have a finite projective resolution over A

$$0 \to P_k \to \cdots \to P_1 \to P_0 \to M \to 0.$$

- Denote this full subcategory of $\mathrm{Hol}(A)$ by $\mathrm{hol}(A)$.

A catch

- Unlike in Pali's case, the algebraic category $\mathrm{hol}(\mathcal{O}(V))$ is too big to model $\mathrm{coh}\, V$ even for projective spaces $V = \mathbb{P}^n_{\mathbb{C}}$.
- For a connection $\nabla_M \colon M \to M \otimes_{\mathcal{O}(V)} \Omega^{(0,1)}$, $m \in M$ and $s \in \mathcal{O}(V)$, we have

$$\nabla_M(ms) = \nabla_M(m)s + m\bar{\partial}(s).$$

  If $v \colon M \to M \otimes_{\mathcal{O}(V)} \Omega^{(0,1)}$ is a homomorphism of $\mathcal{O}(V)$-modules, then $\nabla'_M = \nabla_M + v$ is again a connection.
- In this way, we can construct flat connections with infinite dimensional space of holomorphic global sections

$$\Gamma(M, \nabla'_M) := \ker \nabla'_M.$$

- This never happens for a coherent sheaf over a projective variety!

Differential coherent sheaves

- Classically, if V is a projective variety, then each $\mathscr{F} \in \operatorname{coh} V$ has a presentation of the form

$$(\mathscr{L}^{\otimes t_1})^{n_1} \to (\mathscr{L}^{\otimes t_0})^{n_0} \to \mathscr{F} \to 0$$

($\mathscr{L} \in \operatorname{coh} V$ an ample line bundle).

- For $\mathcal{O}_q(\mathrm{Gr}_{n,r})$, we have a unique quantization for line bundles

$$\nabla_{\mathscr{L}_{n,q}} \colon L_{n,q} \longrightarrow L_{n,q} \otimes_{\mathcal{O}_q(\mathrm{Gr}_{n,r})} \Omega_q^{(0,1)}.$$

(Ó Buachalla and Mrozinski, 2017).

- So we can define the category $\operatorname{coh}_q^{\overline{\partial}} \mathrm{Gr}_{n,r}$ of differential coherent sheaves as the subcategory of $\mathrm{Hol}(\mathcal{O}_q(\mathrm{Gr}_{n,r}))$ consisting of the connections $\nabla_M \colon M \to M \otimes_A \Omega^{(0,1)}$ admitting a presentation

$$\begin{array}{ccccccc}
L_{t_1,q}^{n_1} & \longrightarrow & L_{t_0}^{n_0} & \longrightarrow & M & \longrightarrow & 0 \\
\downarrow \nabla & & \downarrow \nabla & & \downarrow \nabla & & \\
L_{t_1,q}^{n_1} \otimes_{\mathcal{O}_q(\mathrm{Gr}_{n,r})} & \longrightarrow & L_{t_0,q}^{n_0} \otimes_{\mathcal{O}_q(\mathrm{Gr}_{n,r})} & \longrightarrow & M \otimes_{\mathcal{O}_q(\mathrm{Gr}_{n,r})} & \longrightarrow & 0
\end{array}$$

Comparison: algebraic and differential sheaves

- For $\mathcal{O}_q(\mathrm{Gr}_{n,r})$, we have
  1. Algebraic coherent sheaves $\mathrm{coh}_q \mathrm{Gr}_{n,r} = \mathrm{mod}^{\mathbb{Z}} S_q(\mathrm{Gr}_{n,r}) / \mathrm{mod}_0^{\mathbb{Z}} S_q(\mathrm{Gr}_{n,r})$, and
  2. Differential coherent sheaves $\mathrm{coh}_q^{\overline{\partial}} \mathrm{Gr}_{n,r} = \{\nabla_M \colon M \to M \otimes_A \Omega^{(0,1)}\}$.
- The aim is to show that the categories are equivalent.
- For this we need that certain cohomologies vanish in $\mathrm{coh}_q^{\overline{\partial}} \mathrm{Gr}_{n,r}$. More precisely, we focus on cohomologies of the dg $\Omega^{(0,\bullet)}$-module

$$0 \to M \to M \otimes_A \Omega^{(0,1)} \to M \otimes_A \Omega^{(0,2)} \to \cdots$$

  which we obtain by Leibniz rule because ∇_M is flat (Dolbeault cohomology).

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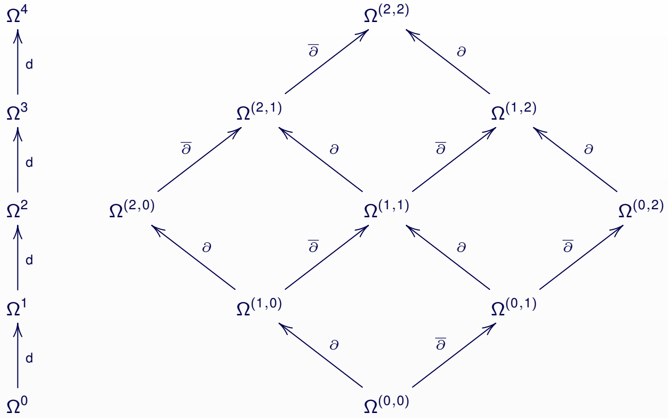
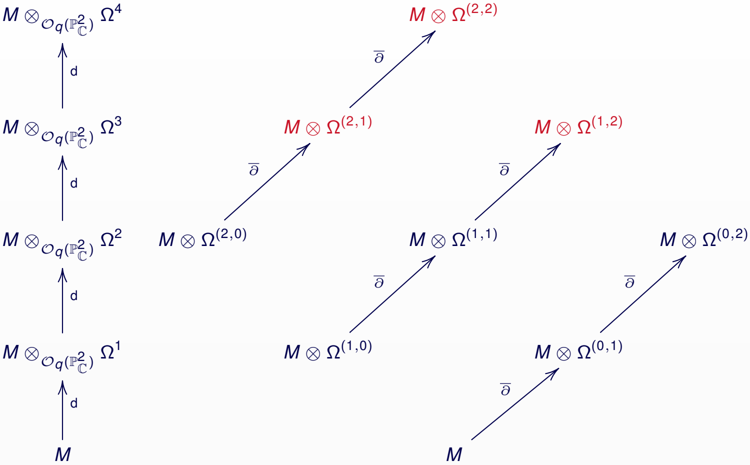
Complex structures

- The complex structure on (quantized or not) $\mathbb{P}^2_{\mathbb{C}}$:



- If $(\nabla_M\colon M \to M \otimes_A \Omega^{(0,1)}) \in \mathrm{Hol}(A)$, we tensor over the dg $\Omega^{(0,\bullet)}$-module with the diamond. Example for $A = \mathcal{O}_q(\mathbb{P}^2_{\mathbb{C}})$:



- Kodaira vanishing (under extra assumptions!)

Theorem (Ó Buachalla, Š., van Roosmalen)

Suppose we have a (non-commutative) Kähler differential calculus (such as the one for $\mathcal{O}_q(\mathrm{Gr}_{n,r})$) and let (M, ∇_M) be a positive Hermitian vector bundle. Then $H^{(a,b)}(M) = 0$ for all $a + b > d$, where d is the dimension of the calculus.

- The non-commutative Kähler structure is defined via a closed real central form $\kappa \in \Omega^{(1,1)}$ such that $L = \kappa \wedge -$ induces isomorphisms $L^{n-k} : \Omega^k \to \Omega^{2n-k}$ for each k.

Theorem (Krutov, Ó Buachalla, Strung)

The line bundles $\nabla_{\mathscr{L}_{t,q}} : L_{t,q} \to L_{t,q} \otimes_{\mathcal{O}_q(\mathrm{Gr}_{n,r})} \Omega^{(0,1)}$ over $\mathcal{O}_q(\mathrm{Gr}_{n,r})$ are positive (= ample) for $t > 0$.

- For quantum Grassmannians, we have the following version of the Bott-Borel-Weil theorem:

Theorem (Ó Buachalla, Š., van Roosmalen)

For $t \geq 0$ and the line bundle $\nabla_{\mathscr{L}_{t,q}} : L_{t,q} \to L_{t,q} \otimes_{\mathcal{O}_q(\mathrm{Gr}_{n,r})} \Omega^{(0,1)}$, we have

$$H^0(L_{t,q}) = V(t\varpi_r) \quad \text{and} \quad H^i(L_{t,q}) = 0 \text{ for all } i > 0.$$

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Abstract ample sequences

Theorem (Artin and Zhang 1994, Polishchuk 2005)

Suppose that $\mathcal{A}$ is an abelian category. Suppose further that we have fixed object $\mathcal{O}_\mathcal{A}$ (an abstract structure sheaf) and an autoequivalence $(1)\colon \mathcal{A} \to \mathcal{A}$ (an abstract twist functor), such that:

1. *$\mathcal{O}_\mathcal{A}$ is noetherian and $\mathrm{Hom}_\mathcal{A}(\mathcal{O}_\mathcal{A})$ is a noetherian $\mathrm{End}_\mathcal{A}(\mathcal{O}_\mathcal{A})$-module for each $M \in \mathcal{A}$.*
2. *For each $M \in \mathcal{A}$, there are integers $t_1, t_2, \dots, t_m$ and an epimorphism $\bigoplus_{i=1}^m \mathcal{O}_\mathcal{A}(-t_i) \twoheadrightarrow M$.*
3. *For each epimorphism $M \twoheadrightarrow N$ in $\mathcal{A}$, there is an integer n_0 such that for every $n \geq n_0$, the map*

$$\mathrm{Hom}_\mathcal{A}(\mathcal{O}_\mathcal{A}, M(n)) \to \mathrm{Hom}_\mathcal{A}(\mathcal{O}_\mathcal{A}, N(n))$$

   is surjective.

Then $\mathcal{A} \simeq \mathrm{mod}^\mathbb{Z} S(\mathcal{A})/\mathrm{mod}_0^\mathbb{Z} S(\mathcal{A})$ for $S(\mathcal{A}) = \bigoplus_{n=0}^\infty \mathrm{Hom}(\mathcal{O}_\mathcal{A}, \mathcal{O}_\mathcal{A}(n))$ (an abstract homogeneous coordinate ring).

- Now we just put everything together.
- For $\mathcal{A} = \mathrm{coh}_q^{\overline{\partial}} \mathrm{Gr}_{n,r}$, the abstract structure sheaf will be $\mathcal{O}_{\mathcal{A}} = (\overline{\partial} \colon \mathcal{O}_q(\mathrm{Gr}_{n,r}) \to \Omega^{(0,1)})$ and we construct a twist functor such that $\mathcal{O}_{\mathcal{A}}(1) = (\nabla_{\mathscr{L}_{1,q}} \colon L_{1,q} \to L_{1,q} \otimes_{\mathcal{O}_q(\mathrm{Gr}_{n,r})} \Omega^{(0,1)})$.
- Now we apply to Bott-Borel-Weil theorem for quantized Grassmannians to obtain

Theorem (Ó Buachalla, Š., van Roosmalen)

The categories $\mathrm{coh}_q \mathrm{Gr}_{n,r} = \mathrm{mod}^{\mathbb{Z}} S_q(\mathrm{Gr}_{n,r}) / \mathrm{mod}_0^{\mathbb{Z}} S_q(\mathrm{Gr}_{n,r})$ *and* $\mathrm{coh}_q^{\overline{\partial}} \mathrm{Gr}_{n,r} = \{\nabla_M \colon M \to M \otimes_A \Omega^{(0,1)}\}$ *are equivalent via*

$$\mathrm{coh}_q^{\overline{\partial}} \mathrm{Gr}_{n,r} \xrightarrow{\Gamma_*} \mathrm{coh}_q \mathrm{Gr}_{n,r},$$

$$(\nabla_M \colon M \to M \otimes_{\mathcal{O}_q(\mathrm{Gr}_{n,r})} \Omega^{(0,1)}) \longmapsto \bigoplus_{n \in \mathbb{Z}} \mathrm{Hom}_{\mathrm{coh}_q^{\overline{\partial}}}(\mathcal{O}_q(\mathrm{Gr}_{n,r}), M(n)).$$

Dolbeault vs. sheaf cohomology

- The Bott-Borel-Weil theorem implies more.
- For each $\nabla_M \colon M \to M \otimes_{\mathcal{O}_q(\mathrm{Gr}_{n,r})} \Omega^{(0,1)}$, we can apply two cohomology theories:
  1. The Dolbeault cohomology—as before, from the complex

$$0 \to M \to M \otimes_A \Omega^{(0,1)} \to M \otimes_A \Omega^{(0,2)} \to \cdots$$

  2. The intrinsic cohomology in the abelian category $\mathrm{coh}_q^{\overline{\partial}}\mathrm{Gr}_{n,r}$:

$$\mathrm{Ext}^n_{\mathrm{coh}_q^{\overline{\partial}}\mathrm{Gr}_{n,r}}(\mathcal{O}_q(\mathrm{Gr}_{n,r}), M)$$

  (abstract sheaf cohomology).

Finiteness for cohomology

Theorem (Ó Buachalla, Š., van Roosmalen)

For each coherent sheaf $\nabla_M \colon M \to M \otimes_{\mathcal{O}_q(\mathrm{Gr}_{n,r})} \Omega^{(0,1)}$ *over a quantum Grassmannian and for each* $n \geq 0$*, the two cohomologies are isomorphic:*

1. $H^n(0 \to M \to M \otimes_A \Omega^{(0,1)} \to M \otimes_A \Omega^{(0,2)} \to \cdots)$ *and*
2. $\mathrm{Ext}^n_{\mathrm{coh}_q^{\overline{\partial}} \mathrm{Gr}_{n,r}}(\mathcal{O}_q(\mathrm{Gr}_{n,r}), M)$.

Corollary

The Dolbeault cohomology of a coherent sheaf is finite dimensional over $\mathbb{C}$*.*

Thank you for your attention!