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Noncommutative algebraic geometry based on quantum flag manifolds

Part I.

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Affine varieties

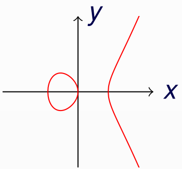
- Let $\mathbb{C}$ be the field of complex numbers and $n \geq 1$.
- A complex affine variety $V \subseteq \mathbb{C}^n$ is just the solution set of a system of polynomial equations, i.e.

$$V = \{P \in \mathbb{C}^n \mid f_i(P) = 0 \text{ for each } i \in I\},$$

where

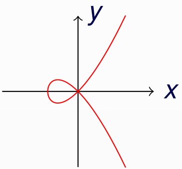
$$f_i \in \mathbb{C}[x_1, x_2, \ldots, x_n] \text{ for each } i \in I.$$

- The real part of $V \subseteq \mathbb{C}^2$ may look like this



$y^2 - x(x-1)(x+1)$

(smooth)



$y^2 - x^2(x+1)$

(singular)

Maps between varieties

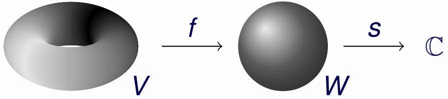
- A map $f\colon V \to W$ of affine varieties ($V \subseteq \mathbb{C}^n$ and $W \subseteq \mathbb{C}^\ell$) if polynomial if there exist $f_1, f_2, \dots, f_\ell \in \mathbb{C}[x_1, x_2, \dots, x_n]$ such that

$$f(P) = \big(f_1(P), f_2(P), \dots, f_\ell(P)\big) \text{ for each } P \in V.$$

- If $V \subseteq \mathbb{C}^n$ is an affine variety, the coordinate ring $\mathbb{C}[V]$ of V is the set of all polynomial maps $f\colon V \to \mathbb{C}$.
- $\mathbb{C}[V]$ is a $\mathbb{C}$-algebra with pointwise operations and as such $\mathbb{C}[V] \cong \mathbb{C}[x_1, x_2, \dots, x_n]/\{f \text{ such that } f|_V \equiv 0\}$ ($\mathbb{C}[V]$ is a finitely generated $\mathbb{C}$-algebra).

Maps control affine varieties

- To each polynomial map $f\colon V \to W$ we may naturally assign a homomorphism of $\mathbb{C}$-algebras $f^*\colon \mathbb{C}[W] \to \mathbb{C}[V]$ given by $f^*(s) = s \circ f$:



- Fact: This assignment induces a bijection between
  1. polynomial maps $V \to W$ and
  2. $\mathbb{C}$-algebra homomorphisms $\mathbb{C}[W] \to \mathbb{C}[V]$.
- A reformulation: There is a full embedding of categories

$$\text{Varieties}_{\mathbb{C}} \longrightarrow (\text{Alg}_{\mathbb{C}})^{\text{op}}.$$

- Hilbert's Nullstellensatz tells us what the image is: These are precisely finitely generated $\mathbb{C}$-algebras R which are reduced: $(\forall\, s \in R)(\forall n \geq 1)(s^n = 0 \implies s = 0)$.
- Analogy with Gelfand-Naimark, $X \leftrightarrow C(X)$ (X compact Hausdorff topological space, $C(X)$ the C^*-algebra of continuous maps $X \to \mathbb{C}$).

The dictionary between algebra and geometry

Affine geometry	Algebra
points of V	maps of $\mathbb{C}$-algebras $\mathbb{C}[V] \to \mathbb{C}$
Cartesian product $V \times W$	tensor product $\mathbb{C}[V] \otimes \mathbb{C}[W]$
affine algebraic groups (such as SL_n)	commutative Hopf algebras
$(\mu\colon G \times G \to G, 1_G \in G)$	$(\mathbb{C}[G] \xrightarrow{\Delta} \mathbb{C}[G] \otimes \mathbb{C}[G], \mathbb{C}[G] \xrightarrow{\varepsilon} \mathbb{C})$

Theorem (Serre, 1955)

For a complex affine variety V, there is a bijective correspondence between

1. *algebraic vector bundles $p\colon E \to V$ and*
2. *certain finitely generated projective $\mathbb{C}[V]$-modules (i.e. direct summands of free $\mathbb{C}[V]$-modules $\mathbb{C}[V]^n$, $n \geq 1$).*

The bijection assigns to a vector bundle p its $\mathbb{C}[V]$-module of sections

$$P = \{s\colon V \to E \text{ polynomial map} \mid p \circ s = 1_V\}.$$

(Quasi-)coherent sheaves

- Problem: Vector bundles do not form an abelian category. More concretely, the image of a map of vector bundles

$$\begin{array}{ccc} E_1 & \xrightarrow{\quad f \quad} & E_2 \\ & {\scriptstyle p_1} \searrow \quad \swarrow {\scriptstyle p_2} & \\ & V & \end{array}$$

  may not be a vector bundle (the ranks of f may differ between fibers).
- Morally, the category of coherent sheaves $\operatorname{coh} V$ is the smallest abelian category containing $\operatorname{Vect} V$. Dictionary:

Affine geometry	Algebra
vector bundles over V	fin. gen. proj. $\mathbb{C}[V]$-modules
coherent sheaves on V	all fin. gen. $\mathbb{C}[V]$-modules
quasi-coherent sheaves on V	all $\mathbb{C}[V]$-modules

- Algebraic principle: If we want to understand properties of a ring R, it is a good idea to study the category of R-modules.

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Projective varieties

- We can define similarly projective algebraic varieties. Projective space:
$$\mathbb{P}^n_{\mathbb{C}} = \{(a_0 : a_1 : \cdots : a_n) \mid (\exists i)(a_i \neq 0)\}.$$
- A complex projective variety $V \subseteq \mathbb{P}^n_{\mathbb{C}}$ is the solution set of a system of homogeneous polynomial equations,
$$V = \{(a_0 : a_1 : \cdots : a_n) \in \mathbb{P}^n_{\mathbb{C}} \mid f_i(a_0, a_1, \ldots, a_n) = 0 \text{ for each } i \in I\}.$$
Here: A polynomial $f \in \mathbb{C}[x_0, x_1, \ldots, x_n]$ is homogeneous if all non-zero terms have the same total degree.
- Similarly, we can take the ideal
$$I(V) = (f \text{ homogeneous} | f_V \equiv 0) \subseteq \mathbb{C}[x_0, x_1, \ldots, x_n]$$
and the homogeneous coordinate ring $S(V) := \mathbb{C}[x_0, x_1, \ldots, x_n]/I(V)$.
- Warning: The elements $f \in S(V)$ typically do not define functions $S(V) \to \mathbb{C}$. Conceptual problem: No holomorphic non-constant maps $\mathbb{P}^1_{\mathbb{C}} \to \mathbb{C}$ by Liouville's theorem!

Regular functions

- Observation: If V is a projective variety and $f, g \in \mathbb{C}[x_0, x_1, \dots, x_n]$ homogeneous of the same degree, then

$$(a_0 : a_1 : \cdots : a_n) \longmapsto \frac{f(a_0, a_1, \dots, a_n)}{g(a_0, a_1, \dots, a_n)} \qquad (*)$$

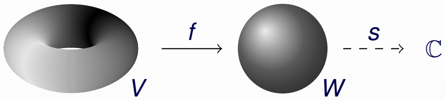
  defines a partial function $V \dashrightarrow \mathbb{C}$.
- Zariski topology on V: the closed sets are the algebraic subsets of V.
- A function $f\colon U \to \mathbb{C}$, $U \subseteq V$ Zariski open, is regular if it is Zariski locally of the form $(*)$.
- What structure should a projective variety actually carry?
- A ringed space is a pair $(V, \mathcal{O}_V)$ such that V is a topological space and $\mathcal{O}_V$ is a sheaf of rings:
  1. for each $U \subseteq V$ we have a ring $\mathcal{O}_V(U)$,
  2. for each $U' \subseteq U \subseteq V$ we have a homomorphism

$$\mathrm{res}^U_{U'} : \mathcal{O}_V(U) \to \mathcal{O}_V(U'),$$

  3. subject to certain axioms.

  (For complex varieties, we have a sheaf of $\mathbb{C}$-algebras!)

- A homomorphism of projective varieties is if a map $f\colon V \to W$ which is Zariski locally computed by ratios of homogeneous polynomials.
- Formally: f is a homomorphisms of varieties if
  1. f is Zariski continuous, and
  2. For each $s \in \mathcal{O}_W(U)$, we have $s \circ f \in \mathcal{O}_V(f^{-1}(U))$.



Related example

If M is a smooth real manifold, M has a structure of ringed space with

$$\mathcal{O}_M(U) = \{s\colon M \to \mathbb{R} \mid s \text{ smooth}\}.$$

A map $f\colon M \to N$ of smooth manifolds is smooth if and only if it satisfies (1) and (2) above.

Vector bundles and Serre's theorem

- If V is a projective variety and $p\colon E \to V$ is an algebraic vector bundle, E might have no non-zero global sections.
- We should consider sections over open subsets $U \subseteq V$:

$$\mathscr{V}(U) = \{s\colon U \to E \mid f \circ s = 1_U\}.$$

- Each $\mathscr{V}(U)$ is an $\mathcal{O}_V(U)$-module, and restrictions $\mathrm{res}^U_{U'}\colon \mathscr{V}(U) \to \mathscr{V}(U')$ are compatible with the module structure.
- Serre, 1955: There is a bijection between
  1. algebraic vector bundles $p\colon E \to V$ and
  2. certain sheaves of $\mathcal{O}_V$-modules such that Zariski locally, $\mathscr{V}(U)$ is a finitely generated projective $\mathcal{O}_V(U)$-module.
- The category of vector bundles can be extended to an abelian category:

$$\mathrm{Vect}\, V \subseteq \mathrm{coh}\, V \subseteq \mathrm{Qcoh}\, V.$$

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Example: Grassmannians

- The set $\mathrm{Gr}_{n,r}$ of r-dimensional vector subspaces of $\mathbb{C}^n$ naturally forms a subset of a projective space via the embedding

$$\iota\colon \mathrm{Gr}_{n,r} \longrightarrow \mathbb{P}_{\mathbb{C}}^{\binom{n}{r}-1},$$
$$V = \langle v_1, v_2, \dots, v_r \rangle \longmapsto \langle v_1 \wedge v_2 \wedge \cdots \wedge v_r \rangle.$$

  (We fix a basis of $\Lambda^r \mathbb{C}^n$ and assign to V its Plücker coordinates.)
- The image of ι is well-known to be a the zero set of quadratic homogeneous polynomials, e.g.

$$\mathrm{Gr}_{4,2} = \{(a_{12} : a_{13} : a_{14} : a_{23} : a_{24} : a_{34}) \in \mathbb{P}^5_{\mathbb{C}} \mid a_{12}a_{34} - a_{13}a_{24} + a_{14}a_{23}\}.$$

Grassmannians as flag varieties

- Representation-theoretic point of view: $\Lambda^r \mathbb{C}^n$ is naturally a representation of $\mathfrak{sl}_n$; it is the r$^{\text{th}}$ fundamental representation $V(\varpi_r)$.
- The image of $\iota \colon \mathrm{Gr}_{n,r} \to \mathbb{P}_{\mathbb{C}}^{\binom{n}{r}-1}$ is identified with the orbit $SL_n \cdot v$ of a highest weight vector $v \in V(\varpi_r)$ and the homogeneous coordinate ring is explicitly given as

$$S(\mathrm{Gr}_{n,r}) \cong \bigoplus_{k=0}^{\infty} V(k\varpi_r)^*.$$

- This generalizes to all flag manifolds F: They are complex projective varieties given by quadratic homogeneous polynomials with the coordinate ring of the form

$$S(F) \cong \bigoplus_{k=0}^{\infty} V(k\lambda)^*.$$

where λ is the sum of the fundamental weights for F.