Jan Šťovíček

Noncommutative algebraic geometry based on quantum flag manifolds

Part I.
(joint with Réamonn Ó Buachalla and Adam-Christiaan van Rooijen)

39th Winter School Geometry and Physics, Srní, January 14th, 2019
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1. Affine algebraic geometry
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Let $\mathbb{C}$ be the field of complex numbers and $n \geq 1$. A complex affine variety $V \subseteq \mathbb{C}^n$ is just the solution set of a system of polynomial equations, i.e.

$$V = \{ P \in \mathbb{C}^n \mid f_i(P) = 0 \text{ for each } i \in I \},$$

where

$$f_i \in \mathbb{C}[x_1, x_2, \ldots, x_n] \text{ for each } i \in I.$$

The real part of $V \subseteq \mathbb{C}^2$ may look like this

\[
\begin{align*}
&y^2 - x(x-1)(x+1) \quad \text{(smooth)} \\
&y^2 - x^2(x+1) \quad \text{(singular)}
\end{align*}
\]
A map $f: V \to W$ of affine varieties ($V \subseteq \mathbb{C}^n$ and $W \subseteq \mathbb{C}^\ell$) if polynomial if there exist $f_1, f_2, \ldots, f_\ell \in \mathbb{C}[x_1, x_2, \ldots, x_n]$ such that

$$f(P) = (f_1(P), f_2(P), \ldots, f_\ell(P)) \text{ for each } P \in V.$$

If $V \subseteq \mathbb{C}^n$ is an affine variety, the coordinate ring $\mathbb{C}[V]$ of $V$ is the set of all polynomial maps $f: V \to \mathbb{C}$.

$\mathbb{C}[V]$ is a $\mathbb{C}$-algebra with pointwise operations and as such $\mathbb{C}[V] \cong \mathbb{C}[x_1, x_2, \ldots, x_n]/\{f \text{ such that } f|_V \equiv 0\}$ ($\mathbb{C}[V]$ is a finitely generated $\mathbb{C}$-algebra).
Maps control affine varieties

- To each polynomial map $f : V \to W$ we may naturally assign a homomorphism of $\mathbb{C}$-algebras $f^* : \mathbb{C}[W] \to \mathbb{C}[V]$ given by $f^*(s) = s \circ f$:

- Fact: This assignment induces a bijection between
  1. polynomial maps $V \to W$ and
  2. $\mathbb{C}$-algebra homomorphisms $\mathbb{C}[W] \to \mathbb{C}[V]$.

- A reformulation: There is a full embedding of categories

\[ \text{Varieties}_\mathbb{C} \longrightarrow (\text{Alg}_\mathbb{C})^{\text{op}}. \]

- Hilbert’s Nullstellensatz tells us what the image is: These are precisely finitely generated $\mathbb{C}$-algebras $R$ which are reduced:

\[ (\forall s \in R)(\forall n \geq 1)(s^n = 0 \implies s = 0). \]

- Analogy with Gelfand-Naimark, $X \leftrightarrow C(X)$ ($X$ compact Hausdorff topological space, $C(X)$ the $C^*$-algebra of continuous maps $X \to \mathbb{C}$).
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<td>affine algebraic groups (such as $SL_n$)</td>
<td>commutative Hopf algebras</td>
</tr>
<tr>
<td>$(\mu: G \times G \rightarrow G, 1_G \in G)$</td>
<td>$(\mathbb{C}[G] \xrightarrow{\Delta} \mathbb{C}[G] \otimes \mathbb{C}[G], \mathbb{C}[G] \xrightarrow{\varepsilon} \mathbb{C})$</td>
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Theorem (Serre, 1955)

For a complex affine variety $V$, there is a bijective correspondence between

1. algebraic vector bundles $p: E \rightarrow V$ and
2. certain finitely generated projective $\mathbb{C}[V]$-modules (i.e. direct summands of free $\mathbb{C}[V]$-modules $\mathbb{C}[V]^n$, $n \geq 1$).

The bijection assigns to a vector bundle $p$ its $\mathbb{C}[V]$-module of sections

$$P = \{ s: V \rightarrow E \text{ polynomial map} \mid p \circ s = 1_V \}.$$
(Quasi-)coherent sheaves

- **Problem:** Vector bundles do not form an abelian category. More concretely, the image of a map of vector bundles

\[ E_1 \xrightarrow{f} E_2 \]

\[ \begin{array}{c}
p_1 \\
\downarrow \\
V \\
\downarrow \\
p_2
\end{array} \]

may not be a vector bundle (the ranks of \( f \) may differ between fibers).

- Morally, the category of coherent sheaves \( \text{coh} V \) is the smallest abelian category containing \( \text{Vect} V \). Dictionary:

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- **Algebraic principle:** If we want to understand properties of a ring \( R \), it is a good idea to study the category of \( R \)-modules.
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Noncommutative flags, part I.
We can define similarly projective algebraic varieties. Projective space:

\[ \mathbb{P}^n_{\mathbb{C}} = \{ (a_0 : a_1 : \cdots : a_n) | (\exists i)(a_i \neq 0) \} \].

A complex projective variety \( V \subseteq \mathbb{P}^n_{\mathbb{C}} \) is the solution set of a system of homogeneous polynomial equations,

\[ V = \{ (a_0 : a_1 : \cdots : a_n) \in \mathbb{P}^n_{\mathbb{C}} | f_i(a_0, a_1, \ldots, a_n) = 0 \text{ for each } i \in I \} \].

Here: A polynomial \( f \in \mathbb{C}[x_0, x_1, \ldots, x_n] \) is homogeneous if all non-zero terms have the same total degree.

Similarly, we can take the ideal

\[ I(V) = (f \text{ homogeneous} | f_V \equiv 0) \subseteq \mathbb{C}[x_0, x_1, \ldots, x_n] \]

and the homogeneous coordinate ring

\[ S(V) := \mathbb{C}[x_0, x_1, \ldots, x_n]/I(V) \].

**Warning:** The elements \( f \in S(V) \) typically do not define functions \( S(V) \rightarrow \mathbb{C} \). Conceptual problem: No holomorphic non-constant maps \( \mathbb{P}^1_{\mathbb{C}} \rightarrow \mathbb{C} \) by Liouville's theorem!
Observation: If $V$ is a projective variety and $f, g \in \mathbb{C}[x_0, x_1, \ldots, x_n]$ homogeneous of the same degree, then

$$(a_0 : a_1 : \cdots : a_n) \mapsto -\frac{f(a_0, a_1, \ldots, a_n)}{g(a_0, a_1, \ldots, a_n)}$$

defines a partial function $V \rightarrow \mathbb{C}$.

Zariski topology on $V$: the closed sets are the algebraic subsets of $V$.

A function $f: U \rightarrow \mathbb{C}$, $U \subseteq V$ Zariski open, is regular if it is Zariski locally of the form $(*)$.

What structure should a projective variety actually carry?

A ringed space is a pair $(V, \mathcal{O}_V)$ such that $V$ is a topological space and $\mathcal{O}_V$ is a sheaf of rings:

1. for each $U \subseteq V$ we have a ring $\mathcal{O}_V(U)$,
2. for each $U' \subseteq U \subseteq V$ we have a homomorphism $\text{res}_{U'}^U: \mathcal{O}_V(U) \rightarrow \mathcal{O}_V(U')$,
3. subject to certain axioms.

(For complex varieties, we have a sheaf of $\mathbb{C}$-algebras!)
Homomorphisms of projective varieties

A homomorphism of projective varieties is a map \( f: V \rightarrow W \) which is Zariski locally computed by ratios of homogeneous polynomials.

Formally: \( f \) is a homomorphism of varieties if

1. \( f \) is Zariski continuous, and
2. For each \( s \in \mathbb{O}_W(U) \), we have \( s \circ f \in \mathbb{O}_V(f^{-1}(U)) \).

Related example

If \( M \) is a smooth real manifold, \( M \) has a structure of a ringed space with

\[
\mathbb{O}_M(U) = \{ s: M \rightarrow \mathbb{R} \mid s \text{ smooth} \}.
\]

A map \( f: M \rightarrow N \) of smooth manifolds is smooth if and only if it satisfies (1) and (2) above.
If \( V \) is a projective variety and \( p: E \to V \) is an algebraic vector bundle, \( E \) might have no non-zero global sections.

We should consider sections over open subsets \( U \subseteq V \):

\[
\mathcal{V}(U) = \{ s: U \to E \mid f \circ s = 1_U \}.
\]

Each \( \mathcal{V}(U) \) is an \( \mathcal{O}_V(U) \)-module, and restrictions \( \text{res}^U_{U'}: \mathcal{V}(U) \to \mathcal{V}(U') \) are compatible with the module structure.

**Serre, 1955:** There is a bijection between

1. algebraic vector bundles \( p: E \to V \) and
2. certain sheaves of \( \mathcal{O}_V \)-modules such that Zariski locally, \( \mathcal{V}(U) \) is a finitely generated projective \( \mathcal{O}_V(U) \)-module.

The category of vector bundles can be extended to an abelian category:

\[
\text{Vect} \, V \subseteq \text{coh} \, V \subseteq \text{Qcoh} \, V.
\]
1. Affine algebraic geometry
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The set $\text{Gr}_{n,r}$ of $r$-dimensional vector subspaces of $\mathbb{C}^n$ naturally forms a subset of a projective space via the embedding

$$\iota : \text{Gr}_{n,r} \longrightarrow \mathbb{P}_{\mathbb{C}}^{\binom{n}{r}-1},$$

$$V = \langle v_1, v_2, \ldots, v_r \rangle \longmapsto \langle v_1 \wedge v_2 \wedge \cdots \wedge v_r \rangle.$$

(We fix a basis of $\Lambda^r \mathbb{C}^n$ and assign to $V$ its Plücker coordinates.)

The image of $\iota$ is well-known to be a the zero set of quadratic homogeneous polynomials, e.g.

$$\text{Gr}_{4,2} = \{(a_{12} : a_{13} : a_{14} : a_{23} : a_{24} : a_{34}) \in \mathbb{P}_{\mathbb{C}}^5 \mid a_{12}a_{34} - a_{13}a_{24} + a_{14}a_{23}\}.$$
Grassmannians as flag varieties

- Representation-theoretic point of view: $\Lambda^r \mathbb{C}^n$ is naturally a representation of $\mathfrak{sl}_n$; it is the $r$th fundamental representation $V(\varpi_r)$.

- The image of $\iota : \text{Gr}_{n,r} \to \mathbb{P}_{\mathbb{C}}^{n-1}$ is identified with the orbit $\text{SL}_n \cdot v$ of a highest weight vector $v \in V(\varpi_r)$ and the homogeneous coordinate ring is explicitly given as

$$S(\text{Gr}_{n,r}) \cong \bigoplus_{k=0}^{\infty} V(k\varpi_r)^*.$$ 

- This generalizes to all flag manifolds $F$: They are complex projective varieties given by quadratic homogeneous polynomials with the coordinate ring of the form

$$S(F) \cong \bigoplus_{k=0}^{\infty} V(k\lambda)^*.$$ 

where $\lambda$ is the sum of the fundamental weights for $F$. 
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Noncommutative algebraic geometry based on quantum flag manifolds

Part II.
(joint with Réamonn Ó Buachalla and Adam-Christiaan van Roosmalen)

39th Winter School Geometry and Physics, Srní, January 16th, 2019
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1 Coherent sheaves on projective varieties

2 Quantized homogeneous rings of flags

3 Relation to the Heckенberger-Kolb calculus
Let $V \subseteq \mathbb{P}_\mathbb{C}^n$ and

$$S(V) = \mathbb{C}[x_0, x_1, \ldots, x_n]/(f \text{ homogeneous, } f|_V \equiv 0)$$

be its homogeneous coordinate ring. Then

$$S(V) = \bigoplus_{n=0}^{\infty} S(V)_n$$

is naturally $\mathbb{Z}$-graded.

**Question:** We know that the elements of $S(V)$ are not functions on $V$. What are they?

The homogeneous parts $S(V)_n, n \geq 0$ are global sections of certain line bundles $\mathcal{L}_n$.

So every projective variety is the set of zeros of sections in line bundles.
The tautological bundle

There is an important line bundle over $\mathbb{P}_\mathbb{C}^n$, the tautological bundle $\mathcal{O}_{\mathbb{P}_\mathbb{C}^n}(1)$.

It is dual to $\mathcal{O}_{\mathbb{P}_\mathbb{C}^n}(-1) \subseteq \mathcal{O}_{\mathbb{P}_\mathbb{C}^n}^{n+1}$, whose the fiber over $(a_0 : a_1 : \cdots : a_n)$ is the line $\langle a_0, a_1, \ldots, a_n \rangle \subseteq \mathbb{C}^{n+1}$.

If $\iota : V \subseteq \mathbb{P}_\mathbb{C}^n$, consider the restricted line bundle $\mathcal{L} := \iota^* \mathcal{O}_{\mathbb{P}_\mathbb{C}^n}(1)$. This is an example of what is called ample (algebraic geometry) or positive (in the context of Kähler manifolds) line bundle.

Fact: $S(V) \cong \bigoplus_{n=0}^{\infty} \Gamma(V, \mathcal{L}^\otimes n)$. The homogeneous coordinate ring is the direct sum of global sections of tensor powers of $\mathcal{L}$. 
The twist functor: If $\mathcal{F} \in \text{Qcoh} \, V$ and $n \in \mathbb{Z}$, put

$$\mathcal{F}(n) := \mathcal{F} \otimes_{\mathcal{O}_V} L^\otimes n$$

(that is, $\mathcal{F}(n)(U) := \mathcal{F}(U) \otimes_{\mathcal{O}_V(U)} L(U)^\otimes n$ on $U$ in an open basis of $V$).

The graded module associated to a sheaf: If $\mathcal{F} \in \text{Qcoh} \, V$, we put

$$\Gamma_{\ast}(\mathcal{F}) := \bigoplus_{n \in \mathbb{Z}} \Gamma(\mathcal{F}(n)) = \bigoplus_{n \in \mathbb{Z}} \text{Hom}((L^\ast)^\otimes n, \mathcal{F}).$$

Example: $\Gamma_{\ast}(\mathcal{O}_V) \cong S(V)$.

**Theorem (Serre, 1955)**

1. The functor $\Gamma_{\ast}: \text{Qcoh} \, V \to \text{Mod}_{\mathbb{Z}} S(V)$ is fully faithful.
2. $\Gamma_{\ast}$ has an exact left adjoint $Q: \text{Mod}_{\mathbb{Z}} S(V) \to \text{Qcoh} \, V$ which satisfies a universal property:
   $$\text{Qcoh} \, V = \text{Mod}_{\mathbb{Z}} S(V) / \text{Mod}_{0} S(V) \text{ (Serre quotient)}.$$
3. Similarly, $\text{coh} \, V = \text{mod}_{\mathbb{Z}} S(V) / \text{mod}_{0} S(V)$. 
Homogeneous coordinate rings of Grassmannians

- We have $SL_n/P \cong Gr_{n,r}$, where

$$P = \begin{pmatrix} P_r & Q \\ 0 & P_{n-r} \end{pmatrix},$$

where $P_r \in M_r(\mathbb{C})$ and $P_{n-r} \in M_{n-r}(\mathbb{C})$. The bijection sends the coset $UP$, $U = (u_{ij})_{i,j=1}^n \in SL_n$ to the linear hull of the first $r$ columns of $U$.

- If we view $Gr_{n,r} \subseteq \mathbb{P}_{\mathbb{C}}^{\binom{n}{r}-1}$ via the Plücker embedding, the quotient map $SL_n \to Gr_{n,r}$ sends $U = (u_{ij})_{i,j=1}^n$ to a point with homogeneous coordinates $\sum_\sigma (-1)^{\text{sgn}(\sigma)} u_{\sigma(i_1),1} u_{\sigma(i_2),2} \cdots u_{\sigma(i_r),r}$, one for each sequence $i_1 < i_2 < \cdots < i_r$.

- In terms of coordinate rings, this shows that $S(Gr_{n,r})$ coincides with the subring

$$\mathbb{C} \left[ \sum_\sigma (-1)^{\text{sgn}(\sigma)} u_{\sigma(i_1),1} u_{\sigma(i_2),2} \cdots u_{\sigma(i_r),r} \mid i_1 < i_2 < \cdots < i_r \right] \subseteq \mathbb{C}[SL_n].$$
In terms of coordinate rings, this shows that \( S(\text{Gr}_n,r) \) coincides with the subring

\[
\mathbb{C} \left[ \sum_{\sigma} (-1)^{\text{sgn}(\sigma)} u_{\sigma(i_1),1} u_{\sigma(i_2),2} \cdots u_{\sigma(i_r),r} \mid i_1 < i_2 < \cdots < i_r \right] \subseteq \mathbb{C}[\text{SL}_n].
\]

We have \( \mathbb{C}[\text{SL}_n] = U(\mathfrak{sl}_n)^\circ \) \((\cdot)^\circ \) is the Hopf dual).

Quantum deformation: We can deform \( \mathbb{C}[\text{SL}_n] \) to \( U_q(\mathfrak{sl}_n)^\circ \) and define \( S_q[\text{Gr}_n,r] \) as the subring

\[
\mathbb{C} \left[ \sum_{\sigma} (-q)^{\ell(\sigma)} u_{\sigma(i_1),1} u_{\sigma(i_2),2} \cdots u_{\sigma(i_r),r} \mid i_1 < i_2 < \cdots < i_r \right] \subseteq U_q(\mathfrak{sl}_n)^\circ.
\]

Representation-theoretic perspective: Again \( S_q[\text{Gr}_n,r] \cong \bigoplus_{k=0}^{\infty} V(k\varpi_r)^* \), where \( V(k\varpi_r) \) is the corresponding representation of \( U_q(\mathfrak{sl}_n) \).
One can do the same for all flags (Soibelman 1992, Taft and Towber 1991, Lakshmibai and Reshetikin 1992, Braveman 1994, ...).

Let $\mathfrak{g}$ be a complex semisimple Lie algebra, $G$ the corresponding complex simply connected algebraic group and $P$ a parabolic subgroup. Then the flag $F = G/P$ is a projective variety and

$$\bigoplus_{k=0}^{\infty} V(k\lambda)^* \cong S_q(F) \subseteq U_q(\mathfrak{g})^\circ,$$

where $\lambda$ is the sum of fundamental weights for $F$ and $V(k\lambda)$ are the corresponding finite dimensional representations of $U_q(\mathfrak{g})$.

One can define a quantization for the category of coherent sheaves: $\text{coh}_q F := \text{mod}^{\mathbb{Z}} S_q(F)/\text{mod}_0^{\mathbb{Z}} S_q(F)$.

This is an abelian category and we can, for instance, define and study the analogue of the sheaf cohomology as well as other algebraic properties.
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2. Quantized homogeneous rings of flags
3. Relation to the Heckenberger-Kolb calculus
Aim: Relate the quantized algebraic and differential geometry.

We have $SU_n \subseteq SL_n$, where

1. $SL_n$ is a complex affine algebraic group,
2. $SU_n$ is a real compact Lie group but it is also a real algebraic group!

Rings of functions in place:

1. For $SL_n$ we have the complex coordinate ring $\mathbb{C}[SL_n]$,
2. For $SU_n$ we have the hierarchy

$$C(SU_n) \supseteq C^\infty(SU_n) \supseteq \mathcal{O}(SU_n)$$

where $\mathcal{O}(SU_n)$ is the ring of polynomial functions $s: SU_n \to \mathbb{C}$ of real algebraic varieties.

3. The magic here: $\mathbb{C}[SL_n] \cong \mathcal{O}(SU_n)$
   (via the restriction of $s: SL_n \to \mathbb{C}$ to $SU_n$).
A “cultural” problem:

1. In differential geometry, a compact complex manifold is a real manifold with an extra structure (flat connection $\bar{\partial}: \mathcal{C}^{\infty}(V) \to \Omega^{(0,1)}$).

2. In algebraic geometry, one usually encounters only polynomial or rational (so holomorphic) functions.

To relate the two, we need a meeting point of (1) and (2).

We have $\text{Gr}_{n,r} \cong SL_n/P \cong SU_n/L$, where

$$P = \begin{pmatrix} P_r & Q \\ 0 & P_{n-r} \end{pmatrix} \quad \text{and} \quad L = P \cap SU_n = \begin{pmatrix} L_r & 0 \\ 0 & L_{n-r} \end{pmatrix}.$$ 

Now:

1. The expression $\text{Gr}_{n,r} \cong SL_n/P$ allows to view the Grassmannian as a projective complex algebraic variety.

2. The expression $\text{Gr}_{n,r} \cong SU_n/L$ allows to view the Grassmannian as an affine real algebraic variety.

The meeting point: Try to view a complex algebraic variety $V$ as a real algebraic variety with a “complex structure” (a flat connection $\bar{\partial}: \mathcal{O}(V) \to \Omega^{(0,1)}$).
Dolbeault dg algebra

- If $V$ is a complex manifold, we have the Dolbeault complex:

$$
0 \longrightarrow C^\infty(V) \xrightarrow{\overline{\partial}} \Omega^{(0,1)} \xrightarrow{\overline{\partial}} \Omega^{(0,2)} \xrightarrow{\overline{\partial}} \ldots
$$

- We can wedge forms ($\wedge : \Omega^{(0,i)} \otimes \Omega^{(0,j)} \longrightarrow \Omega^{(0,i+j)}$). Then $\Omega^{(0,\bullet)} = \bigoplus \Omega^{(0,i)}$ is a $\mathbb{Z}$-graded associative algebra over $\mathbb{C}$.

- Moreover, we have the graded Leibniz rule:

$$
\overline{\partial}(\omega_i \wedge \omega_j) = \overline{\partial}(\omega_i) \wedge \omega_j + (-1)^i \omega_i \wedge \overline{\partial}(\omega_j)
$$

for each $\omega_i \in \Omega^{(0,i)}$ and $\omega_j \in \Omega^{(0,j)}$. In other words, $(\Omega^{(0,\bullet)}(V), \wedge, \overline{\partial})$ is a differential graded (dg) algebra.

- If $V = \text{Gr}_{n,r} = SU_n/L$, then

$$
\mathcal{O}(\text{Gr}_{n,r}) \subseteq C^\infty(\text{Gr}_{n,r}) \subseteq C(\text{Gr}_{n,r})
$$

and $\mathcal{O}(\text{Gr}_{n,r})$ is dense with respect to $\| - \|_\infty$.

- Now, the Dolbeault dg algebra for $\text{Gr}_{n,r}$ does restrict to real algebraic sections:

$$
0 \longrightarrow \mathcal{O}(\text{Gr}_{n,r}) \xrightarrow{\overline{\partial}} \Omega^{(0,1)}_{\text{alg}} \xrightarrow{\overline{\partial}} \Omega^{(0,2)}_{\text{alg}} \xrightarrow{\overline{\partial}} \ldots
$$
The Dolbeault dg algebra for $\text{Gr}_{n,r}$ does restrict to real algebraic sections:

$$
\begin{align*}
0 & \longrightarrow \mathcal{O}(\text{Gr}_{n,r}) \overset{\overline{\partial}}{\longrightarrow} \Omega_{\text{alg}}^{(0,1)} \overset{\overline{\partial}}{\longrightarrow} \Omega_{\text{alg}}^{(0,2)} \overset{\overline{\partial}}{\longrightarrow} \ldots \\
\end{align*}
$$

and can be quantized:

$$
\begin{align*}
0 & \longrightarrow \mathcal{O}_q(\text{Gr}_{n,r}) \overset{\overline{\partial}}{\longrightarrow} \Omega^{(0,1)}_q \overset{\overline{\partial}}{\longrightarrow} \Omega^{(0,2)}_q \overset{\overline{\partial}}{\longrightarrow} \ldots \\
\end{align*}
$$

$$
((\Omega^{(0,\cdot)}_q(\text{Gr}_{n,r}), \wedge, \overline{\partial}) \text{ is a dg algebra again}).
$$

If we impose some more natural conditions on $\Omega^{(0,\cdot)}_q(\text{Gr}_{n,r})$, it is unique (Heckenberger and Kolb, 2006)!

In fact, Heckenberger and Kolb quantized the Dolbeault dg algebra for all compact Hermitian symmetric flags.
The Koszul-Malgrange theorem

Theorem (Koszul and Malgrange, 1958)

Let $V$ be a compact complex manifold. Then there is a bijective correspondence between

1. holomorphic vector bundles $p: E \to V$ and
2. smooth complex vector bundles equipped with a flat connection $\nabla_E: \Gamma^\infty(E) \to \Gamma^\infty(E) \otimes_{\mathcal{C}^\infty(V)} \Omega^{(0,1)}$, where

\[
\Gamma^\infty(E) = \{ s: V \to E \text{ smooth map} \mid p \circ s = 1_V \}.
\]

The holomorphic sections of $E$ are precisely $\nabla_E$.

- By a version of the Serre-Swan theorem, $\Gamma^\infty(E)$ is a finitely generated projective $\mathcal{C}^\infty(V)$-module.
- Define quantized algebraic vector bundles over $\text{Gr}_{n,r}$ as flat connections $\nabla: P \to P \otimes_{\mathcal{O}_q(\text{Gr}_{n,r})} \Omega^{(0,1)}_q$, where $P$ is a finitely generated projective $\mathcal{O}_q(\text{Gr}_{n,r})$-module.
Recall: On $\text{Gr}_{n,r} = SU_{n+1}/L$, we have only one reasonable quantized Dolbeault dg algebra $(\Omega_q^{(0,\bullet)}, \wedge, \overline{\partial})$.

Since $\text{Gr}_{n,r}$ is homogeneous, one can use representation theory of $\mathfrak{l}$ to construct quantum deformations $L_{n,q}$ of tensor powers $L^\otimes n$ of the tautological bundle $L$.

That is, there are finitely generated projective $L_{n,q}$ are finitely generated projective $\mathcal{O}_q(\text{Gr}_{n,r})$-modules and certain flat connections, unique by Ó Buachalla and Mrozinski,

$$\nabla \mathcal{L}_{n,q} : L_{n,q} \rightarrow L_{n,q} \otimes \mathcal{O}_q(\text{Gr}_{n,r}) \Omega_q^{(0,1)}.$$

**Theorem (Ó Buachalla and Mrozinski, 2017)**

*For each $n \geq 0$, We have $S_q(\text{Gr}_{n,r})_n \cong \ker \nabla \mathcal{L}_{n,q}$.*

So the holomorphic sections of line bundles based on the Heckenberger-Kolb calculus and the Koszul-Malgrange theorem agree with the older “naive” construction of the quantized coordinate ring.
Noncommutative algebraic geometry based on quantum flag manifolds
Part III.
(joint with Réamonn Ó Buachalla and Adam-Christiaan van Roosmalen)

39th Winter School Geometry and Physics, Srní, January 18th, 2019
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1. Coherent sheaves from the differential point of view

2. Cohomology of differential line bundles

3. Comparison of the algebraic/differential approaches
Recall: If $V$ is a compact complex manifold, then a holomorphic vector bundle $p: E \to V$ can be equivalently given via a flat connection

$$\nabla_E: \Gamma^\infty(E) \to \Gamma^\infty(E) \otimes_{C^\infty(V)} \Omega^{(0,1)}.$$  

(Koszul and Malgrange, 1958).

There is a generalization for coherent sheaves over $V$:

**Theorem (Pali, 2006)**

Given a compact complex manifold $V$ and the sheaf $\mathcal{O}_V^\infty$ of smooth complex-valued functions, there is a bijective correspondence between

1. analytic coherent sheaves on $V$ and
2. flat connections $\nabla_G: \mathcal{G} \to \mathcal{G} \otimes \mathcal{O}_V^\infty \Omega^{(0,1)}$, where the sheaf $\mathcal{O}_V^\infty$-modules locally admits a resolution

$$0 \to (\mathcal{O}_V^\infty|_U)^{n_k} \to \cdots \to (\mathcal{O}_V^\infty|_U)^{n_1} \to (\mathcal{O}_V^\infty|_U)^{n_0} \to \mathcal{G}|_U \to 0.$$
### Theorem (Pali, 2006)

Given a compact complex manifold $V$ and the sheaf $\mathcal{O}_V^\infty$ of smooth complex-valued functions, there is a bijective correspondence between

1. analytic coherent sheaves on $V$ and
2. flat connections $\nabla_{\mathcal{G}} : \mathcal{G} \to \mathcal{G} \otimes \mathcal{O}_V^\infty \Omega^{(0,1)}$, where the sheaf $\mathcal{O}_V^\infty$-modules locally admits a resolution

$$0 \to (\mathcal{O}_V^\infty|_U)^{n_k} \to \cdots \to (\mathcal{O}_V^\infty|_U)^{n_1} \to (\mathcal{O}_V^\infty|_U)^{n_0} \to \mathcal{G}|_U \to 0.$$  

- If $V$ embeds into a projective space, then $V$ is a smooth complex projective algebraic variety (Chow, 1949).
- In that case, the categories of analytic and algebraic coherent sheaves are equivalent (Serre’s GAGA Theorem, 1956).
- Pali actually proves that $\text{coh} V$ is equivalent to the category of the flat connections as above.
This motivated Beggs and Smith (2012) to define an abelian category $\text{Hol}(A)$ for a non-commutative complex structure $(\Omega^\bullet(A), d = \partial + \overline{\partial})$ (e.g. $A = \mathcal{O}_q(\text{Gr}_{n,r})$ as before):

The objects are flat connections $\nabla_M : M \to M \otimes_A \Omega^{(0,1)}$ and the morphisms are given by $f : M \to N$ such that

\[
\begin{array}{ccc}
M & \xrightarrow{\nabla_M} & M \otimes_A \Omega^{(0,1)} \\
\downarrow f & & \downarrow f \otimes \Omega^{(0,1)} \\
N & \xrightarrow{\nabla_N} & N \otimes_A \Omega^{(0,1)}
\end{array}
\]

First approximation to the differential description of the category of coherent sheaves: require that $M$ have a finite projective resolution over $A$

\[
0 \to P_k \to \cdots \to P_1 \to P_0 \to M \to 0.
\]

Denote this full subcategory of $\text{Hol}(A)$ by $\text{hol}(A)$. 
Unlike in Pali’s case, the algebraic category $\text{hol}(\mathcal{O}(V))$ is too big to model $\text{coh}V$ even for projective spaces $V = \mathbb{P}^n_{\mathbb{C}}$.

For a connection $\nabla_M : M \rightarrow M \otimes_{\mathcal{O}(V)} \Omega^{0,1}$, $m \in M$ and $s \in \mathcal{O}(V)$, we have

$$\nabla_M(ms) = \nabla_M(m)s + m\overline{\partial}(s).$$

If $\nu : M \rightarrow M \otimes_{\mathcal{O}(V)} \Omega^{0,1}$ is a homomorphism of $\mathcal{O}(V)$-modules, then $\nabla'_M = \nabla_M + \nu$ is again a connection.

In this way, we can construct flat connections with infinite dimensional space of holomorphic global sections

$$\Gamma(M, \nabla'_M) := \ker \nabla'_M.$$

This never happens for a coherent sheaf over a projective variety!
Differential coherent sheaves

- Classically, if $V$ is a projective variety, then each $\mathcal{F} \in \text{coh } V$ has a presentation of the form

\[
(L \otimes t_1)^{n_1} \rightarrow (L \otimes t_0)^{n_0} \rightarrow \mathcal{F} \rightarrow 0
\]

($L \in \text{coh } V$ an ample line bundle).

- For $\mathcal{O}_q(\text{Gr}_n,r)$, we have a unique quantization for line bundles

\[
\nabla_{L_n,q} : L_n,q \rightarrow L_n,q \otimes \mathcal{O}_q(\text{Gr}_n,r) \Omega_q^{0,1}
\]

(Ó Buachalla and Mrozinski, 2017).

- So we can define the category $\text{coh}_q\overline{\partial}\text{Gr}_n,r$ of differential coherent sheaves as the subcategory of $\text{Hol}(\mathcal{O}_q(\text{Gr}_n,r))$ consisting of the connections $\nabla_M : M \rightarrow M \otimes_A \Omega^{(0,1)}$ admitting a presentation

\[
\begin{array}{ccccccccc}
L_{t_1,q}^{n_1} & \rightarrow & L_{t_0}^{n_0} & \rightarrow & M & \rightarrow & 0 \\
\downarrow \nabla & & \downarrow \nabla & & \downarrow \nabla & & \\
L_{t_1,q}^{n_1} \otimes \mathcal{O}_q(\text{Gr}_n,r) & \rightarrow & L_{t_0,q}^{n_0} \otimes \mathcal{O}_q(\text{Gr}_n,r) & \rightarrow & M \otimes \mathcal{O}_q(\text{Gr}_n,r) & \rightarrow & 0
\end{array}
\]
For $\mathcal{O}_q(\operatorname{Gr}_{n,r})$, we have

1. **Algebraic coherent sheaves**
   
   \[ \text{coh}_q \operatorname{Gr}_{n,r} = \text{mod}^\mathbb{Z} S_q(\operatorname{Gr}_{n,r})/\text{mod}^\mathbb{Z}_0 S_q(\operatorname{Gr}_{n,r}), \]

2. **Differential coherent sheaves**

   \[ \text{coh}_q^{\partial} \operatorname{Gr}_{n,r} = \{ \nabla_M : M \to M \otimes_A \Omega^{(0,1)} \}. \]

The aim is to show that the categories are equivalent.

For this we need that certain cohomologies vanish in $\text{coh}_q^{\partial} \operatorname{Gr}_{n,r}$. More precisely, we focus on cohomologies of the dg $\Omega^{(0,\bullet)}$-module

\[ 0 \to M \to M \otimes_A \Omega^{(0,1)} \to M \otimes_A \Omega^{(0,2)} \to \ldots \]

which we obtain by Leibniz rule because $\nabla_M$ is flat (Dolbeault cohomology).
1. Coherent sheaves from the differential point of view

2. Cohomology of differential line bundles

3. Comparison of the algebraic/differential approaches
The complex structure on (quantized or not) $\mathbb{P}^2_{\mathbb{C}}$: 
Complex structure and ‘holomorphic’ connections

If \( \nabla_M : M \rightarrow M \otimes_A \Omega^{(0,1)} \) \( \in \text{Hol}(A) \), we tensor over the dg \( \Omega^{(0,\bullet)} \)-module with the diamond. Example for \( A = \mathcal{O}_q(\mathbb{P}^2_C) \):

Kodaira vanishing (under extra assumptions!)

\[
\begin{align*}
M \otimes_{\mathcal{O}_q(\mathbb{P}^2_C)} \Omega^4 & \quad \rightarrow \quad M \otimes \Omega^{(2,2)} \\
M \otimes_{\mathcal{O}_q(\mathbb{P}^2_C)} \Omega^3 & \quad \rightarrow \quad M \otimes \Omega^{(2,1)} \\
M \otimes_{\mathcal{O}_q(\mathbb{P}^2_C)} \Omega^2 & \quad \rightarrow \quad M \otimes \Omega^{(1,2)} \\
M \otimes_{\mathcal{O}_q(\mathbb{P}^2_C)} \Omega^1 & \quad \rightarrow \quad M \otimes \Omega^{(0,2)} \\
M & \quad \rightarrow \quad M \otimes \Omega^{(0,1)}
\end{align*}
\]
Theorem (Ó Buachalla, Š., van Roosmalen)

Suppose we have a (non-commutative) Kähler differential calculus (such as the one for \( \mathcal{O}_q(\text{Gr}_{n,r}) \)) and let \((M, \nabla_M)\) be a positive Hermitian vector bundle. Then \(H^{(a,b)}(M) = 0\) for all \(a + b > d\), where \(d\) is the dimension of the calculus.

- The non-commutative Kähler structure is defined via a closed real central form \(\kappa \in \Omega^{(1,1)}\) such that \(L = \kappa \wedge -\) induces isomorphisms \(L^{n-k} : \Omega^k \to \Omega^{2n-k}\) for each \(k\).

Theorem (Krutov, Ó Buachalla, Strung)

The line bundles \(\nabla \mathcal{L}_{t,q} : L_{t,q} \to L_{t,q} \otimes \mathcal{O}_q(\text{Gr}_{n,r}) \Omega^{(0,1)}\) over \(\mathcal{O}_q(\text{Gr}_{n,r})\) are positive (= ample) for \(t > 0\).
For quantum Grassmannians, we have the following version of the Bott-Borel-Weil theorem:

**Theorem (Ó Buachalla, Š., van Roosmalen)**

For $t \geq 0$ and the line bundle $\nabla L_{t,q} : L_{t,q} \to L_{t,q} \otimes O_q(Gr_{n,r}) \Omega^{(0,1)}$, we have

$$H^0(L_{t,q}) = V(t \varpi_r) \quad \text{and} \quad H^i(L_{t,q}) = 0 \text{ for all } i > 0.$$
1. Coherent sheaves from the differential point of view
2. Cohomology of differential line bundles
3. Comparison of the algebraic/differential approaches
Abstract ample sequences

Theorem (Artin and Zhang 1994, Polishchuk 2005)

Suppose that \( \mathcal{A} \) is an abelian category. Suppose further that we have fixed object \( \mathcal{O}_\mathcal{A} \) (an abstract structure sheaf) and an autoequivalence (1): \( \mathcal{A} \to \mathcal{A} \) (an abstract twist functor), such that:

1. \( \mathcal{O}_\mathcal{A} \) is noetherian and \( \text{Hom}_\mathcal{A}(\mathcal{O}_\mathcal{A}) \) is a noetherian \( \text{End}_\mathcal{A}(\mathcal{O}_\mathcal{A}) \)-module for each \( M \in \mathcal{A} \).
2. For each \( M \in \mathcal{A} \), there are integers \( t_1, t_2, \ldots, t_m \) and an epimorphism \( \bigoplus_{i=1}^m \mathcal{O}_\mathcal{A}(-t_i) \to M \).
3. For each epimorphism \( M \twoheadrightarrow N \) in \( \mathcal{A} \), there is an integer \( n_0 \) such that for every \( n \geq n_0 \), the map

\[
\text{Hom}_\mathcal{A}(\mathcal{O}_\mathcal{A}, M(n)) \to \text{Hom}_\mathcal{A}(\mathcal{O}_\mathcal{A}, N(n))
\]

is surjective.

Then \( \mathcal{A} \cong \text{mod} \mathbb{Z} S(\mathcal{A})/\text{mod}_0 \mathbb{Z} S(\mathcal{A}) \) for \( S(\mathcal{A}) = \bigoplus_{n=0}^{\infty} \text{Hom}(\mathcal{O}_\mathcal{A}, \mathcal{O}_\mathcal{A}(n)) \) (an abstract homogeneous coordinate ring).
Now we just put everything together.

For $\mathcal{A} = \text{coh}_q \text{Gr}_{n,r}$, the abstract structure sheaf will be
$$\mathcal{O}_\mathcal{A} = (\bar{\partial} : \mathcal{O}_q(\text{Gr}_{n,r}) \to \Omega^{(0,1)})$$
and we construct a twist functor such that $\mathcal{O}_\mathcal{A}(1) = (\nabla \mathcal{L}_1 : \mathcal{L}_1 \to \mathcal{L}_1 \otimes \mathcal{O}_q(\text{Gr}_{n,r}) \Omega^{(0,1)})$.

Now we apply to Bott-Borel-Weil theorem for quantized Grassmannians to obtain

**Theorem (Ó Buachalla, Š., van Roosmalen)**

The categories $\text{coh}_q \text{Gr}_{n,r} = \text{mod}^\mathbb{Z} S_q(\text{Gr}_{n,r})/\text{mod}^\mathbb{Z} S_q(\text{Gr}_{n,r})$ and $\text{coh}_{\bar{\partial}} \text{Gr}_{n,r} = \{ \nabla_M : M \to M \otimes_A \Omega^{(0,1)} \}$ are equivalent via

$$\text{coh}_{\bar{\partial}} \text{Gr}_{n,r} \xrightarrow{\Gamma_*} \text{coh}_q \text{Gr}_{n,r},$$

$$(\nabla_M : M \to M \otimes \mathcal{O}_q(\text{Gr}_{n,r}) \Omega^{(0,1)}) \leftrightarrow \bigoplus_{n \in \mathbb{Z}} \text{Hom}_{\text{coh}_q}(\mathcal{O}_q(\text{Gr}_{n,r}), M(n)).$$
The Bott-Borel-Weil theorem implies more.

For each $\nabla_M : M \to M \otimes \mathcal{O}_q(\text{Gr}_{n,r}) \Omega^{(0,1)}$, we can apply two cohomology theories:

1. The Dolbeault cohomology—as before, from the complex

$$0 \to M \to M \otimes_A \Omega^{(0,1)} \to M \otimes_A \Omega^{(0,2)} \to \cdots$$

2. The intrinsic cohomology in the abelian category $\text{coh}^{\overline{q}} \text{Gr}_{n,r}$:

$$\text{Ext}^n_{\text{coh}^{\overline{q}} \text{Gr}_{n,r}} (\mathcal{O}_q(\text{Gr}_{n,r}), M)$$

(abstract sheaf cohomology).
Theorem (Ó Buachalla, Š., van Roosmalen)

For each coherent sheaf $\nabla_M: M \rightarrow M \otimes \mathcal{O}_{q(Gr_{n,r})} \Omega^{(0,1)}$ over a quantum Grassmannian and for each $n \geq 0$, the two cohomologies are isomorphic:

1. $H^n(0 \rightarrow M \rightarrow M \otimes_A \Omega^{(0,1)} \rightarrow M \otimes_A \Omega^{(0,2)} \rightarrow \cdots)$
2. $\operatorname{Ext}^n_{\operatorname{coh}_{qGr_{n,r}}} (\mathcal{O}_{q(Gr_{n,r})}, M)$.

Corollary

The Dolbeault cohomology of a coherent sheaf is finite dimensional over $\mathbb{C}$.

Thank you for your attention!