Derived categories with a view towards Grothendieck duality

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Outline

1. Functors from module categories
2. Derived categories
3. Sheaf cohomology
4. A view towards Grothendieck duality
Let \((R, +, -, 0, \cdot, 1)\) be a commutative ring and let \(\text{Mod}R\) be the category of right \(R\)-modules.

Examples to keep in mind in this talk: \(M\) a manifold/alg. variety over a field \(k\), \(R\) the ring of all smooth/holomorphic/polynomial functions.

\(R\)-modules can encode among others vector bundles over \(M\). More details will follow.
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Rings and modules

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Non-exactness of Hom functors

Given any module $X$, the assignment

$$Y \in \text{Mod}_R \quad \mapsto \quad \text{Hom}_R(X, Y)$$

gives a functor

$$\text{Hom}_R(X, -): \text{Mod}_R \rightarrow \text{Ab}$$

This functor is left-exact, i.e. given a short exact sequence

$$0 \rightarrow K \rightarrow Y \rightarrow C \rightarrow 0$$

of modules, we obtain an exact sequence of groups

$$0 \rightarrow \text{Hom}_R(X, K) \rightarrow \text{Hom}_R(X, Y) \rightarrow \text{Hom}_R(X, C) \rightarrow \ast$$

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Derived functors

- **Classical answer:** derived functors.
- Given a left exact functor
  \[ F : \text{Mod}_R \to \text{Ab}, \]
  (e.g. \( F = \text{Hom}_R(X, -) \)), there is a canonical way to produce a series of functors
  \[ R^1F, R^2F, R^3F, \ldots : \text{Mod}_R \to \text{Ab} \]
  (the right derived functors of \( F \)) such that starting with a short exact sequence of \( R \)-modules
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- The derived category $\mathcal{D}(R)$ of the module category $\text{Mod}R$ provides a flexible language for homological algebra and the right framework for working with derived functors.
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- It is a complicated object, though. It contains infinitely many copies of $\text{Mod}_R$ as full subcategories:
Construction of the derived category

- We start the category $\mathbb{C}(R)$ of cochain complexes over $\text{Mod} R$. An object $X$ of $\mathbb{C}(R)$ is a diagram of modules

$$
\ldots \rightarrow X^{-1} \xrightarrow{\partial^{-1}} X^0 \xrightarrow{\partial^0} X^1 \xrightarrow{\partial^1} X^2 \rightarrow \ldots
$$

such that $\partial \circ \partial = 0$.

- Morphisms $f : X \rightarrow Y$ in $\mathbb{C}(R)$ are defined in the obvious way as a commutative diagrams

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\quad \begin{array}{c}
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- Cohomology modules: $H^n(X) = \ker \partial^n / \text{Im} \partial^{n-1}$ for an integer $n$. This is in fact again a functor, $H^n : \mathbb{C}(R) \rightarrow \text{Mod} R$. 

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Jan Šťovíček (Charles University)
The point: We are interested in the cohomology of complexes rather then in the complexes themselves.

That is, if $f : X \to Y$ is a homomorphism of complexes such that

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is an isomorphism for all $n \in \mathbb{Z}$, then $f$ morally should be an isomorphism. Such morphisms are called quasi-isomorphisms.

Such $f$’s are ubiquitous, we have for example for $R = \mathbb{Z}$:

$$\cdots \longrightarrow 0 \longrightarrow \mathbb{Z} \xrightarrow{m} \mathbb{Z} \longrightarrow 0 \longrightarrow \cdots$$

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To summarize so far: We have the category $\mathcal{C}(R)$ of cochain complexes of $R$-modules and the class $\Sigma$ of all quasi-morphisms. We consider quasi-isomorphic complexes as “the same”.

The brutal step: We force the quasi-isomorphisms to become isomorphisms. That is, we formally add an inverse to every $\sigma \in \Sigma$.

Up to some inessential set-theoretical annoyances, one can always do this. The result is by definition the derived category $\mathcal{D}(R) = \mathcal{C}(R)[\Sigma^{-1}]$. It comes together with the canonical “localization” functor

$$Q : \mathcal{C}(R) \longrightarrow \mathcal{D}(R).$$

$Q$ sends every $\sigma \in \Sigma$ to an isomorphism and it is a universal functor with this property.

The hard part is to understand the category we get.
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Taking smaller steps

- We shall perform the passage from $\mathbf{C}(R)$ to $\mathbf{D}(R)$ in two smaller steps.

- Making a morphism invertible can cause another morphism to become zero (if $\sigma \circ \alpha = 0$, then $\alpha$ vanishes after making $\sigma$ invertible).

- First we take the quotient category $\mathbf{K}(R) = \mathbf{C}(R)/\mathcal{I}$, where $\mathcal{I}$ is a two-sided ideal of some (not all) morphisms which must vanish under $Q : \mathbf{C}(R) \to \mathbf{D}(R)$. $\mathcal{I}$ is the class of the so-called null-homotopic morphisms of complexes.

- In the second step, the morphisms from $\Sigma$ (or more precisely their images under $Q : \mathbf{C}(R) \to \mathbf{K}(R)$) are made invertible. That is, we construct $\mathbf{D}(R)$ as $\mathbf{K}(R)[\Sigma^{-1}]$.

- The latter is more tractable since $\Sigma$ is a multiplicative system in $\mathbf{K}(R)$ (unlike in $\mathbf{C}(R)$). In other words, it allows the calculus of left and right fractions.
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- In the second step, the morphisms from $\Sigma$ (or more precisely their images under $Q': \mathbf{C}(R) \to \mathbf{K}(R)$) are made invertible. That is, we construct $\mathbf{D}(R)$ as $\mathbf{K}(R)[\Sigma^{-1}]$.
- The latter is more tractable since $\Sigma$ is a multiplicative system in $\mathbf{K}(R)$ (unlike in $\mathbf{C}(R)$). In other words, it allows the calculus of left and right fractions.
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- We shall perform the passage from $\mathbf{C}(R)$ to $\mathbf{D}(R)$ in two smaller steps.
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Calculus of left fractions

**LF1** If $\sigma, \tau$ are composable morphisms in $\Sigma$

$$X \xrightarrow{\sigma} Y \xrightarrow{\tau} Z$$

then so is $\tau \circ \sigma$. The identity morphisms $1_X : X \rightarrow X$ belong to $\Sigma$ for every object $X$.

**LF2** Given morphisms $\alpha, \sigma$ with $\sigma \in \Sigma$

$$X \xrightarrow{\alpha} Y$$

$$\downarrow \sigma \quad \quad \quad \quad \quad \quad \downarrow \tau$$

$$Z \xrightarrow{\beta} W$$

(i.e. a “right fraction” $\alpha \cdot \sigma^{-1}$), we can form a commutative square with $\tau \in \Sigma$ (i.e. a $\alpha \cdot \sigma^{-1} = \tau^{-1} \cdot \beta$).
Calculus of left fractions

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Calculus of left fractions

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LF2 Given morphisms $\alpha, \sigma$ with $\sigma \in \Sigma$

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\xymatrix{ X \ar[r]^-{\alpha} \ar[d]_-{\sigma} & Y \ar[d]^-{\tau} \\
Z \ar[r]_-{\beta} & W }
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LF3 Let $\alpha$ be a morphism. If there is $\sigma \in \Sigma$ such that

$$\xymatrix{ X \ar[r]^\sigma & Y \ar[r]^\alpha & Z \ar@{=}[l] }$$

composes to zero (i.e. $\alpha$ must become zero in $K(R)[\Sigma^{-1}]$), then there is $\tau \in \Sigma$ such that

$$\xymatrix{ Y \ar[r]^\alpha & Z \ar[r]^\tau & W \ar@{=}[l] }$$

composes to zero.
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$$X \xrightarrow{\sigma} Y \xrightarrow{\alpha} Z$$

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Equality of fractions

Suppose we have two left fractions $\sigma_1^{-1} \cdot \alpha_1$ and $\sigma_2^{-1} \cdot \alpha_2$ between objects $Z$ and $Y$:

![Diagram]

We make $\sigma_1^{-1} \cdot \alpha_1$ and $\sigma_2^{-1} \cdot \alpha_2$ equal provided that we can complete the above diagram in such a way that $\tau \in \Sigma$. 
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Composing morphisms

Suppose we have two composable fractions $\sigma_1^{-1} \cdot \alpha_1$ and $\sigma_2^{-1} \cdot \alpha_2$:

The composition is defined (using LF1) to be the fraction

$$\beta \circ \alpha_1 \quad \text{and} \quad \tau \circ \sigma_2.$$
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Suppose we have two composable fractions $\sigma_1^{-1} \cdot \alpha_1$ and $\sigma_2^{-1} \cdot \alpha_2$:

```
\begin{array}{ccc}
X & \alpha_1 & U \\
\uparrow & & \downarrow \\
Y & \sigma_1 & V \\
\downarrow & & \downarrow \\
Z & \alpha_2 & W \\
\end{array}
```

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```
\begin{array}{ccc}
X & \beta \circ \alpha_1 & W \\
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W & \tau \circ \sigma_2 & Z \\
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```
Composing morphisms

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\begin{array}{ccc}
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\uparrow & & \downarrow \beta \\
Y & \xleftarrow{\sigma_1} & W \\
\downarrow & & \downarrow \tau \\
Z & \xrightarrow{\alpha_2} & V \\
\downarrow & & \downarrow \sigma_2 \\
& \xleftarrow{\lf_{2}} & \\
& & \end{array}
\]

The composition is defined (using $\lf_1$) to be the fraction

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\downarrow & & \downarrow \tau \circ \sigma_2 \\
& & Z. \\
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\]
Construction of the derived category—summary

- $\mathcal{D}(R) = \mathcal{C}(R)[\Sigma^{-1}]$, where $\Sigma$ is the class of quasi-isomorphisms.
- To understand what $\mathcal{D}(R)$ looks like, it is better to take two smaller steps:

$$
\begin{array}{ccc}
\mathcal{C}(R) & \xrightarrow{Q} & \mathcal{D}(R) \\
\text{quotient} & & \text{Ore localization} \\
\downarrow & & \downarrow \\
\mathcal{K}(R) & & \\
\end{array}
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- The category carries an extra important structure: It is a so-called triangulated category. As far as this talk is concerned, this feature is ruthlessly ignored!
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Suppose again that we have a left exact functor $F : \text{Mod} R \rightarrow \text{Ab}$. If we apply $F$ to a complex of $R$-modules, the result is a complex of abelian groups. That is, we naturally get a functor

$$F : \mathbf{C}(R) \rightarrow \mathbf{C}(\text{Ab}).$$

It is essentially for free to push $F$ further to

$$F : \mathbf{K}(R) \rightarrow \mathbf{K}(\text{Ab}).$$

We encounter troubles if we wish to go one step further and construct a functor

$$\mathbf{D}(R) \rightarrow \mathbf{D}(\text{Ab}).$$

Cause: If $\sigma$ is a quasi-isomorphism, $F(\sigma)$ need not be (the non-exactness of $F$ again!)
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Derived functors—continued

- Since we cannot lift $F$ to a functor $\mathbf{D}(R) \to \mathbf{D}(\text{Ab})$ directly, we take the “best” approximation—the total right derived functor

  $$RF: \mathbf{D}(R) \longrightarrow \mathbf{D}(\text{Ab}).$$

- Given a complex $X \in \mathbf{K}(R)$, we define $RF(X)$ indirectly via

  $$\text{Hom}_{\mathbf{D}(\text{Ab})}(-, RF(X)) = \lim_{\to} \text{Hom}_{\mathbf{D}(\text{Ab})}(-, F(C)),$$

  where the colimit is indexed by the comma-category of all quasi-isomorphisms $X \xrightarrow{\sigma} C$. The functor $RF(X)$ is well defined since the comma category has a terminal object $X \xrightarrow{T} i(X)$ (a so-called $K$-injective resolution of $X$). Then in fact $RF(X) = F(i(X))$.

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Outline

1. Functors from module categories
2. Derived categories
3. Sheaf cohomology
4. A view towards Grothendieck duality
Suppose $M$ is a smooth manifold and $U \subseteq M$ be an open subset. Denote by $\mathcal{O}_M(U)$ the ring of all smooth functions $U \to \mathbb{R}$.

Given open $V \subseteq U \subseteq M$, we have the restriction homomorphism of $\mathbb{R}$-algebras:

$$\text{res}_V^U : \mathcal{O}_M(U) \longrightarrow \mathcal{O}_M(V), \quad f \longmapsto f|_V.$$

Fact: The manifold structure on $M$ is completely determined by

1. the topology on $M$ and
2. the collection $\mathcal{O}_M = (\mathcal{O}_M(U), \text{res}_V^U)$ of the $\mathbb{R}$-algebras $\mathcal{O}_M(U)$ together with the restriction homomorphisms $\text{res}_V^U$. $\mathcal{O}_M$ is called the structure sheaf of $M$.

This is an alternative description of a manifold compared to giving an atlas. Especially popular in algebraic geometry.
The structure sheaf

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$$\text{res}^U_V: \mathcal{O}_M(U) \to \mathcal{O}_M(V), \quad f \mapsto f|_V.$$

Fact: The manifold structure on $M$ is completely determined by

1. the topology on $M$ and
2. the collection $\mathcal{O}_M = (\mathcal{O}_M(U), \text{res}^U_V)$ of the $\mathbb{R}$-algebras $\mathcal{O}_M(U)$ together with the restriction homomorphisms $\text{res}^U_V$. $\mathcal{O}_M$ is called the structure sheaf of $M$.

This is an alternative description of a manifold compared to giving an atlas. Especially popular in algebraic geometry.
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Vector bundles

- A vector bundle over $M$ is a surjective morphism of manifolds $\pi : E \to M$ for which there is $n \geq 0$ and an open cover $M = \bigcup_i U_i$ such that for every $i$:

$$
\begin{array}{ccc}
\pi^{-1}(U_i) & \sim & U_i \times \mathbb{R}^n \\
\pi|_{U_i} & \downarrow & \text{proj.} \\
U_i & \downarrow &
\end{array}
$$

- Given $U \subseteq M$ open, denote by

$$
\mathcal{E}(U) = \{ s : U \to \pi^{-1}(U) \mid \pi \circ s = 1_U \},
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the collection of sections over the set $U$. This is naturally an $\mathcal{O}_M(U)$-module, and it is free if $U = U_i$ for some $i$. 

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Derived categories

October 21, 2011 21 / 28
Starting with a vector bundle $\pi : E \to M$, we obtain the following:

1. For each open $U \subseteq M$ an $\mathcal{O}_M(U)$-module $\mathcal{E}(U)$ of sections over $U$;
2. For each open $V \subseteq U \subseteq M$ the restriction maps $\text{res}^U_V : \mathcal{E}(U) \to \mathcal{E}(V), \ s \mapsto s\big|_V$
   compatible with the module structure;
3. The so-called sheaf axiom holds: a section over an open set can be glued together from sections over smaller open sets;
4. Locally, $\mathcal{E}(U)$ is a free module of a fixed finite rank.

Fact: 1 and 2 determine the vector bundle structure of $\pi : E \to M$.

Collections $\mathcal{E} = (\mathcal{E}(U), \text{res}^U_V)$ satisfying 1–3 are called sheaves of $\mathcal{O}_M$-modules. In particular, vector bundles can be viewed as special sheaves of $\mathcal{O}_M$-modules.
Sheaf cohomology

Vector bundles–continued

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Sheaves of $\mathcal{O}_M$-modules

- The category of all sheaves of $\mathcal{O}_M$-modules is denoted by $\text{Mod}\mathcal{O}_M$.
- It is an abelian category—i.e. the notion of a short exact sequence makes sense and it is well-behaved. In fact more is known: it is a so-called Grothendieck category.
- We can form the derived category $\mathbf{D}(\mathcal{O}_M)$ as for usual modules over rings, and we can construct total derived functors of functors $F : \text{Mod}\mathcal{O}_M \to \text{Ab}$.
- The same constructions can be done for complex analytic manifolds and algebraic varieties.
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The same constructions can be done for complex analytic manifolds and algebraic varieties.
Sheaf cohomology

- There is a special functor for which the derived functor is particularly interesting: the global section functor:

\[ \Gamma_M : \text{Mod}\mathcal{O}_M \to \text{Ab}, \quad \mathcal{E} \mapsto \mathcal{E}(M). \]

- Observation \( \Gamma = \text{Hom}_{\mathcal{O}_M}(\mathcal{O}_M, -) \). In particular, \( \Gamma \) is left exact, but need not be exact.

- Sheaf cohomology functors \( H^n_M : \text{Mod}\mathcal{O}_M \to \text{Ab} \) are defined as the right derived functors \( R^n\Gamma_M \). They tell us something about the global geometry of \( M \).

- With our machinery, we are now able to construct the total derived functor

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Outline

1. Functors from module categories
2. Derived categories
3. Sheaf cohomology
4. A view towards Grothendieck duality
Theorem (Serre 1955)

Let \( M \) be a compact connected complex manifold of dimension \( d \geq 0 \). Let \( \Omega^d_M \) be the line bundle of holomorphic \( d \)-forms, and given a vector bundle \( \mathcal{E} \in \text{Mod}\mathcal{O}_M \), denote \( \mathcal{E}^* = \text{Hom}(\mathcal{E}, \Omega^d_M) \).

Then for every \( i \) in the range \( 0 \leq i \leq d \), there is a natural isomorphism

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H^i_M(\mathcal{E})^* \cong H^{d-i}_M(\mathcal{E}^*).
\]
The Serre duality theorem

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Theorem (reformulation, Grothendieck)

Consider the global section functor as a functor

\[ \Gamma_M : \text{Mod}\mathcal{O}_M \longrightarrow \text{Mod}\mathbb{C}. \]

Then a restriction of \( R\Gamma_M \) to a suitable subcategory of \( D(\mathcal{O}_M) \), which contains all vector bundles, has a right adjoint.

Remark

\( \Gamma_M \) is a left exact functor, not right exact. Therefore, \( \Gamma_M \) definitely cannot have a right adjoint, but \( R\Gamma_M \) has!
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Grothendieck duality—continued

- Given a morphism of manifolds $f : M \to N$, there is standard pushforward functor
  $$f_* : \text{Mod}O_M \longrightarrow \text{Mod}O_N$$

- If $N = \{ \star \}$ is a single point, then $f_*$ equals
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- The Grothendieck duality theorem says that in algebraic geometry, some restriction of $Rf_*$ has a right adjoint in much broader generality that for $N = \{ \star \}$. 
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