Derived categories with a view towards Grothendieck duality

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Charles University in Prague

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Outline

Rings and modules

- Let $(R, +, -, 0, \cdot, 1)$ be a commutative ring and let Mod$R$ be the category of right R-modules.
- Examples to keep in mind in this talk: M a manifold/alg. variety over a field k, R the ring of all smooth/holomorphic/polynomial functions

- R-modules can encode among others vector bundles over M. More details will follow ▸ vector bdl's.

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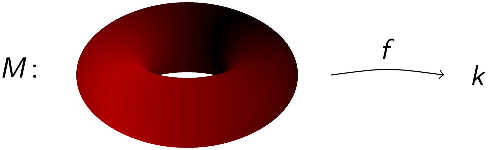

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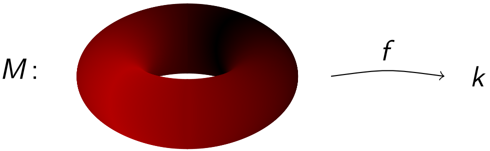

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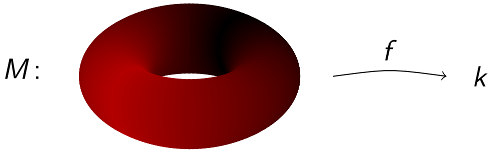

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Non-exactness of Hom functors

- Given any module X, the assignment

$$Y \in \operatorname{Mod} R \quad \longmapsto \quad \operatorname{Hom}_R(X, Y)$$

gives a functor

$$\operatorname{Hom}_R(X, -) \colon \operatorname{Mod} R \longrightarrow \mathrm{Ab}$$

- This functor is left-exact, i.e. given a short exact sequence

$$0 \longrightarrow K \longrightarrow Y \longrightarrow C \longrightarrow 0$$

of modules, we obtain an exact sequence of groups

$$0 \longrightarrow \operatorname{Hom}_R(X, K) \longrightarrow \operatorname{Hom}_R(X, Y) \longrightarrow \operatorname{Hom}_R(X, C) \longrightarrow *$$

- What should be there instead of $*$?

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Derived functors

- Classical answer: derived functors.
- Given a left exact functor

$$F\colon \mathrm{Mod}R \longrightarrow \mathrm{Ab},$$

(e.g. $F = \mathrm{Hom}_R(X, -)$), there is a canonical way to produce a series of functors

$$\mathbf{R}^1F, \mathbf{R}^2F, \mathbf{R}^3F, \dots : \mathrm{Mod}R \longrightarrow \mathrm{Ab}$$

(the right derived functors of F) such that starting with a short exact sequence of R-modules

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we obtain a natural long exact sequence

$$0 \longrightarrow F(K) \longrightarrow F(Y) \longrightarrow F(C) \longrightarrow \mathbf{R}^1F(K) \longrightarrow \mathbf{R}^1F(Y) \longrightarrow$$
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- The derived category $\mathbf{D}(R)$ of the module category $\operatorname{Mod} R$ provides a flexible language for homological algebra and the right framework for working with derived functors.
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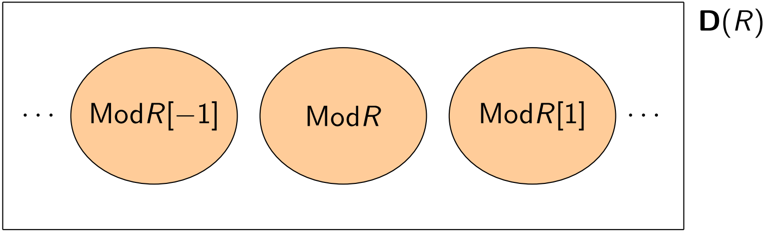

Construction of the derived category

- We start the the category $\mathbf{C}(R)$ of cochain complexes over Mod R. An object X of $\mathbf{C}(R)$ is a diagram of modules

$$\cdots \longrightarrow X^{-1} \xrightarrow{\partial^{-1}} X^0 \xrightarrow{\partial^0} X^1 \xrightarrow{\partial^1} X^2 \longrightarrow \cdots$$

such that $\partial \circ \partial = 0$.

- Morphisms $f: X \to Y$ in $\mathbf{C}(R)$ are defined in the obvious way as a commutative diagrams

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- Cohomology modules: $H^n(X) = \operatorname{Ker} \partial^n / \operatorname{Im} \partial^{n-1}$ for an integer n. This is in fact again a functor, $H^n : \mathbf{C}(R) \to \operatorname{Mod} R$.

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Construction of the derived category—continued

- The point: We are interested in the cohomology of complexes rather then in the complexes themselves.
- That is, if $f : X \to Y$ is a homomorphisms of complexes such that

$$H^n(f)\colon H^n(X) \longrightarrow H^n(Y)$$

is an isomorphism for all $n \in \mathbb{Z}$, then f morally should be an isomorphism. Such morphisms are called quasi-isomorphisms.
- Such f's are ubiquitous, we have for example for $R = \mathbb{Z}$:

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Construction of the derived category—continued

- To summarize so far: We have the category $\mathbf{C}(R)$ of cochain complexes of R-modules and the class Σ of all quasi-morphisms. We consider quasi-isomorphic complexes as "the same".
- The brutal step: We force the quasi-isomorphisms to become isomorphisms. That is, we formally add an inverse to every $\sigma \in \Sigma$.
- Up to some inessential set-theoretical annoyances, one can always do this. The result is by definition the derived category $\mathbf{D}(R) = \mathbf{C}(R)[\Sigma^{-1}]$. It comes together with the canonical "localization" functor

$$Q\colon \mathbf{C}(R) \longrightarrow \mathbf{D}(R).$$

  Q sends every $\sigma \in \Sigma$ to an isomorphism and it is a universal functor with this property.
- The hard part is to understand the category we get.

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Taking smaller steps

- We shall perform the passage from $\mathbf{C}(R)$ to $\mathbf{D}(R)$ in two smaller steps.
- Making a morphism invertible can cause another morphism to become zero (if $\sigma \circ \alpha = 0$, then α vanishes after making σ invertible).
- First we take the quotient category $\mathbf{K}(R) = \mathbf{C}(R)/\mathcal{I}$, where $\mathcal{I}$ is a two-sided ideal of some (not all) morphisms which must vanish under $Q\colon \mathbf{C}(R) \to \mathbf{D}(R)$. $\mathcal{I}$ is the class of the so-called null-homotopic morphisms of complexes.
- In the second step, the morphisms from Σ (or more precisely their images under $Q'\colon \mathbf{C}(R) \twoheadrightarrow \mathbf{K}(R)$) are made invertible. That is, we construct $\mathbf{D}(R)$ as $\mathbf{K}(R)[\Sigma^{-1}]$.
- The latter is more tractable since Σ is a multiplicative system in $\mathbf{K}(R)$ (unlike in $\mathbf{C}(R)$). In other words, it allows the calculus of left and right fractions. skip fractions

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Calculus of left fractions

LF1 If σ, τ are composable morphisms in Σ

$$X \xrightarrow{\sigma} Y \xrightarrow{\tau} Z$$

then so is $\tau \circ \sigma$. The identity morphisms $1_X : X \longrightarrow X$ belong to Σ for every object X.

LF2 Given morphisms α, σ with $\sigma \in \Sigma$

$$\begin{array}{ccc} X & \xrightarrow{\alpha} & Y \\ \sigma \downarrow & & \downarrow \tau \\ Z & \xrightarrow[\beta]{} & W \end{array}$$

(i.e. a “right fraction” $\alpha \cdot \sigma^{-1}$), we can form a commutative square with $\tau \in \Sigma$ (i.e. a $\alpha \cdot \sigma^{-1} = \tau^{-1} \cdot \beta$).

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Equality of fractions

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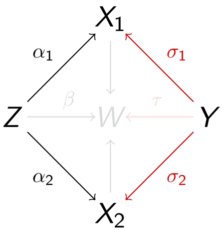

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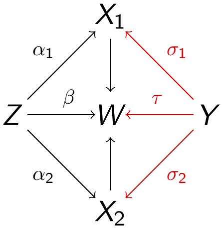

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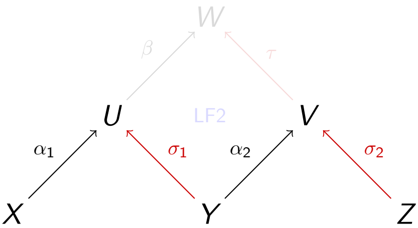

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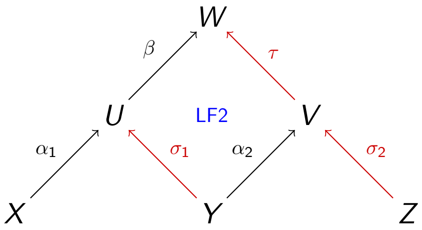

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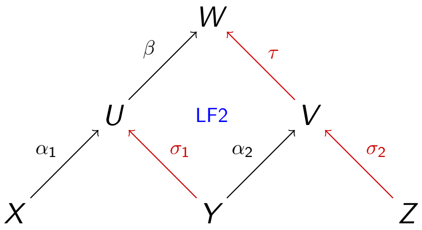

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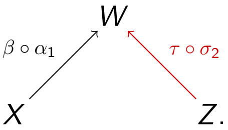

Construction of the derived category—summary

- $\mathbf{D}(R) = \mathbf{C}(R)[\Sigma^{-1}]$, where Σ is the class of quasi-isomorphisms.
- To understand what $\mathbf{D}(R)$ looks like, it is better to take two smaller steps:

$$\begin{array}{ccc} \mathbf{C}(R) & \xrightarrow{Q} & \mathbf{D}(R) \\ & \text{quotient} \searrow \quad \nearrow \text{Ore localization} & \\ & \mathbf{K}(R) & \end{array}$$

- The category carries an extra important structure: It is a so-called triangulated category. As far as this talk is concerned, this feature is ruthlessly ignored!

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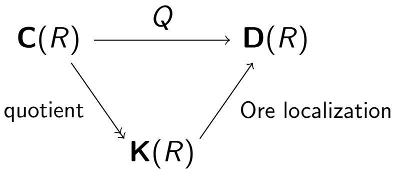

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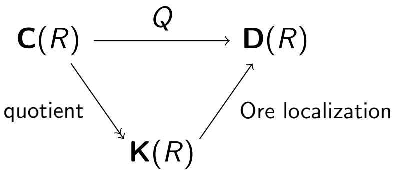

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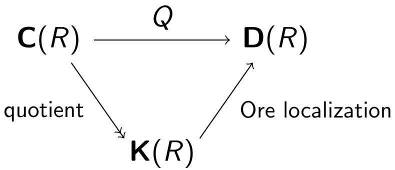

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Derived functors

- Suppose again that we have a left exact functor $F\colon \mathrm{Mod}R \longrightarrow \mathrm{Ab}$. If we apply F to a complex of R-modules, the result is a complex of abelian groups. That is, we naturally get a functor

$$F\colon \mathbf{C}(R) \longrightarrow \mathbf{C}(\mathrm{Ab}).$$

- It is essentially for free to push F further to

$$F\colon \mathbf{K}(R) \longrightarrow \mathbf{K}(\mathrm{Ab}).$$

- We encounter troubles if we wish to go one step further and construct a functor

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- Cause: If σ is a quasi-isomorphism, $F(\sigma)$ need not be (the non-exactness of F again!)

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- Since we cannot lift F to a functor $\mathbf{D}(R) \to \mathbf{D}(\mathrm{Ab})$ directly, we take the “best” approximation—the total right derived functor

$$\mathbf{R}F\colon \mathbf{D}(R) \longrightarrow \mathbf{D}(\mathrm{Ab}).$$

- Given a complex $X \in \mathbf{K}(R)$, we define $\mathbf{R}F(X)$ indirectly via

$$\mathrm{Hom}_{\mathbf{D}(\mathrm{Ab})}(-, \mathbf{R}F(X)) = \varinjlim \mathrm{Hom}_{\mathbf{D}(\mathrm{Ab})}(-, F(C)),$$

where the colimit is indexed by the comma-category of all quasi-isomorphisms $X \xrightarrow{\sigma} C$. The functor $\mathbf{R}F(X)$ is well defined since the comma category has a terminal object $X \xrightarrow{\tau} i(X)$ (a so-called K-injective resolution of X). Then in fact $\mathbf{R}F(X) = F(i(X))$.

- Fact: $H^n \circ \mathbf{R}F \cong \mathbf{R}^n F$ for each $n \geq 0$. That is, the total right derived functor $\mathbf{R}F$ contains all information about the classical right derived functors $\mathbf{R}^n F$!

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Outline

The structure sheaf

- Suppose M is a smooth manifold and $U \subseteq M$ be an open subset. Denote by $\mathcal{O}_M(U)$ the ring of all smooth functions $U \to \mathbb{R}$.
- Given open $V \subseteq U \subseteq M$, we have the restriction homomorphism of $\mathbb{R}$-algebras:

$$\mathrm{res}^U_V \colon \mathcal{O}_M(U) \longrightarrow \mathcal{O}_M(V), \quad f \longmapsto f|_V.$$

- Fact: The manifold structure on M is completely determined by
  1. the topology on M and
  2. the collection $\mathcal{O}_M = (\mathcal{O}_M(U), \mathrm{res}^U_V)$ of the $\mathbb{R}$-algebras $\mathcal{O}_M(U)$ together with the restriction homomorphisms res^U_V. $\mathcal{O}_M$ is called the structure sheaf of M.
- This is an alternative description of a manifold compared to giving an atlas. Especially popular in algebraic geometry.

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Vector bundles

- A vector bundle over M is a surjective morphism of manifolds $\pi\colon E \to M$ for which there is $n \geq 0$ and an open cover $M = \bigcup_i U_i$ such that for every i:

$$\begin{array}{ccc} \pi^{-1}(U_i) & \xrightarrow{\sim} & U_i \times \mathbb{R}^n \\ & {\scriptstyle \pi|_{U_i}} \searrow \quad \swarrow {\scriptstyle \text{proj.}} & \\ & U_i & \end{array}$$

- Given $U \subseteq M$ open, denote by

$$\mathcal{E}(U) = \{s\colon U \to \pi^{-1}(U) \mid \pi \circ s = 1_U\},$$

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- Starting with a vector bundle $\pi\colon E \to M$, we obtain the following:
  1. for each open $U \subseteq M$ an $\mathcal{O}_M(U)$-module $\mathcal{E}(U)$ of sections over U;
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  $$\mathrm{res}^U_V\colon \mathcal{E}(U) \longrightarrow \mathcal{E}(V), \quad s \longmapsto s|_V$$
  compatible with the module structure;
  3. the so-called sheaf axiom holds: a section over an open set can be glued together from sections over smaller open sets;
  4. locally, $\mathcal{E}(U)$ is a free module of a fixed finite rank.
- Fact: 1 and 2 determine the vector bundle structure of $\pi\colon E \to M$.
- Collections $\mathcal{E} = (\mathcal{E}(U), \mathrm{res}^U_V)$ satisfying 1–3 are called sheaves of $\mathcal{O}_M$-modules. In particular, vector bundles can be viewed as special sheaves of $\mathcal{O}_M$-modules.

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Sheaves of $\mathcal{O}_M$-modules

- The category of all sheaves of $\mathcal{O}_M$-modules is denoted by $\mathrm{Mod}\mathcal{O}_M$.
- It is an abelian category—i.e. the notion of a short exact sequence makes sense and it is well-behaved. In fact more is known: it is a so-called Grothendieck category.
- We can form the derived category $\mathbf{D}(\mathcal{O}_M)$ as for usual modules over rings, and we can construct total derived functors of functors $F\colon \mathrm{Mod}\mathcal{O}_M \longrightarrow \mathrm{Ab}$.
- The same constructions can be done for complex analytic manifolds and algebraic varieties.

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Sheaf cohomology

- There is a special functor for which the derived functor is particularly interesting: the global section functor:

$$\Gamma_M \colon \mathrm{Mod}\mathcal{O}_M \longrightarrow \mathrm{Ab}, \quad \mathcal{E} \longmapsto \mathcal{E}(M).$$

- Observation $\Gamma = \mathrm{Hom}_{\mathcal{O}_M}(\mathcal{O}_M, -)$. In particular, Γ is left exact, but need not be exact.
- Sheaf cohomology functors $H^n_M \colon \mathrm{Mod}\mathcal{O}_M \to \mathrm{Ab}$ are defined as the right derived functors $\mathbf{R}^n\Gamma_M$. They tell us something about the global geometry of M.
- With our machinery, we are now able to construct the total derived functor

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Outline

The Serre duality theorem

Theorem (Serre 1955)

Let M be a compact connected complex manifold of dimension $d \geq 0$. Let Ω^d_M be the line bundle of holomorphic d-forms, and given a vector bundle $\mathcal{E} \in \mathrm{Mod}\mathcal{O}_M$, denote $\mathcal{E}^* = \mathcal{H}om(\mathcal{E}, \Omega^d_M)$.

Then for every i in the range $0 \leq i \leq d$, there is a natural isomorphism

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Theorem (reformulation, Grothendieck)

Consider the global section functor as a functor

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Then a restriction of $\mathbf{R}\Gamma_M$ *to a suitable subcategory of* $\mathbf{D}(\mathcal{O}_M)$*, which contains all vector bundles, has a right adjoint.*

Remark

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Grothendieck duality—continued

- Given a morphism of manifolds $f\colon M \to N$, there is standard pushforward functor

$$f_*\colon \mathrm{Mod}\mathcal{O}_M \longrightarrow \mathrm{Mod}\mathcal{O}_N$$

- If $N = \{\star\}$ is a single point, then f_* equals

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- The Grothendieck duality theorem says that in algebraic geometry, some restriction of $\mathbf{R}f_*$ has a right adjoint in much broader generality that for $N = \{\star\}$.

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