Abstract representation theory using Grothendieck derivators

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Charles University in Prague

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Outline

1. Homotopy (co)limits
2. Grothendieck derivators
3. Stability
4. Abstract representation theory

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General setup

Problem

Given a category $\mathcal{C}$ and a class W or morphisms, understand $\mathcal{C}[W^{-1}]$.

Examples

1. $\mathcal{C} = \mathcal{T}op$ (topological spaces), $W =$ weak equivalences (morphisms inducing bijections on all homotopy groups).
2. $\mathcal{C} = \mathrm{C}(\mathcal{A})$, complexes over and abelian category $\mathcal{A}$, $W = \{$homology isomorphisms$\}$.

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Trouble with constructions

Well known problem: $\mathcal{C}[W^{-1}]$ often does not admit good categorical constructions.

Example

If $\mathrm{D}(\mathcal{A}) = \mathrm{C}(\mathcal{A})[\{\text{q.-iso}\}^{-1}]$, then every monomorphism and every epimorphism splits. Therefore, there are not so many interesting limits or colimits.

Usual algebraic solution: Keep some of the information originally contained in $\mathrm{C}(\mathcal{A})$ as an additional structure to $\mathrm{D}(\mathcal{A})$ (triangulated structure, mapping cones).

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Homotopy invariant constructions

Let again $\mathcal{C}$ be a category and W a class of morphism to invert, abstract weak equivalences. $\mathcal{C}$ is usually quite well behaved in that it is complete and cocomplete.

Problem

$$\begin{array}{ccccc} X & \xrightarrow{f} & Y & \xrightarrow{\text{coker}} & C \\ \uparrow{\scriptstyle\sim} & & \uparrow{\scriptstyle\sim} & & \uparrow{\scriptstyle\not\sim} \\ X' & \xrightarrow[f']{} & Y' & \xrightarrow[\text{coker}]{} & C' \end{array}$$

Well known: Often one functorially choose f' which is initial in a suitable sense so that C' is homotopy invariant. This is always possible if $(\mathcal{C}, W)$ has a structure of a Quillen model category. Classical cases:

1. Cofiber sequences of pointed spaces in homotopy theory.
2. Mapping cones of complexes in algebra.

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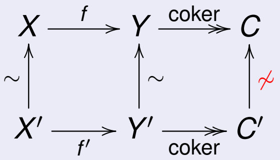

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The moral from the cone construction

- Denote by $[n]$ the category generated by

$$0 \to 1 \to 2 \to \cdots \to n.$$

- If $(\mathcal{C}, W)$ is nice enough (model category), we have for $I = [1]$:

$$\begin{array}{cc} \mathcal{C}^I & \mathcal{C} \\ \mathcal{C}^I[W_I^{-1}] & \mathcal{C}[W^{-1}] \\ (\mathcal{C}[W^{-1}])^I & \end{array}$$

- The functor diag_I is far from being in equivalence. If k is a field and $\mathcal{C} = \mathrm{C}(k)$, then diag_I is essentially the homology functor

$$H_* : \mathrm{D}(k(\cdot \to \cdot)) \longrightarrow \mathrm{Mod}^{\mathbb{Z}} k(\cdot \to \cdot).$$

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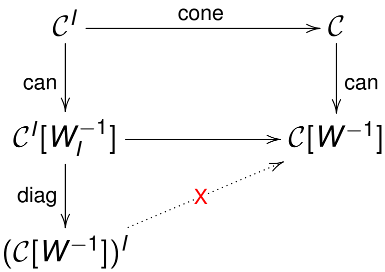

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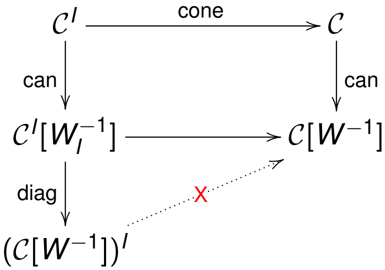

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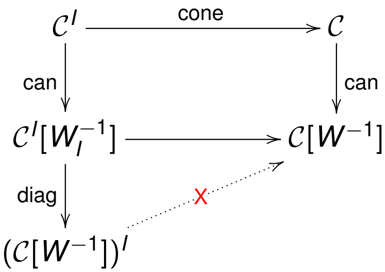

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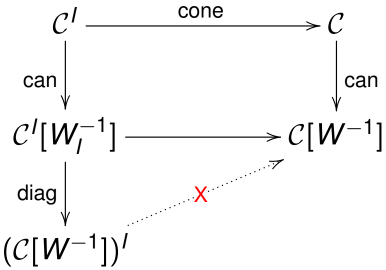

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Expressing (co)limits abstractly

- Let $\mathcal{C}$ be a cocomplete category and $I \in \mathcal{C}at$ a small category. Then we have

$$\mathcal{C}^I \underset{\text{const}}{\overset{\text{colim}}{\rightleftarrows}} \mathcal{C}.$$

- We just derive the adjoint pair of functors!

$$\mathcal{C}[W^{-1}]^I \xleftarrow{\text{diag}} \mathcal{C}^I[W_I^{-1}] \underset{\text{const}}{\overset{\text{hocolim}}{\rightleftarrows}} \mathcal{C}[W^{-1}],$$

where hocolim $=$ **L**colim.

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Prederivators

Idea (Grothendieck, Heller, others)

Given $(\mathcal{C}, W)$, the category of I-shaped diagrams in the homotopy category $\mathcal{C}[W^{-1}]$ contains too little information. We need to remember $\mathcal{C}^I[W_I^{-1}]$ instead, i.e. the homotopy category of I-shaped diagrams.

Definition

A prederivator is a strict 2-functor $\mathscr{D}\colon \mathcal{C}at^{\mathrm{op}} \to \mathrm{CAT}$:

$$\mathscr{D}\colon \quad I \underset{g}{\overset{f}{\rightrightarrows}} J \;\; (\Downarrow \eta) \quad \longmapsto \quad \mathscr{D}(I) \underset{g^*}{\overset{f^*}{\leftleftarrows}} \mathscr{D}(J) \;\; (\Downarrow \eta^*)$$

A derivator is a prederivator satisfying certain axioms (to come) allowing for a well behaved calculus of homotopy Kan extensions.

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Axioms for a derivator

(Der1) $\mathscr{D}(\coprod_j I_j) \xrightarrow{\sim} \prod_j \mathscr{D}(I_j)$ canonically.

(Der2) The diagram functor $\mathrm{diag}\colon \mathscr{D}(I) \to \mathscr{D}(*)^I$ reflects isomorphisms (conservativity axiom).

(Der3) Let $f\colon I \to J$ be a functor in $\mathcal{C}at$. Then the restriction functor f^* has both a left adjoint $f_!$ and a right adjoint f_*:

$$\mathscr{D}(I) \underset{f_*}{\overset{f_!}{\rightrightarrows}} \mathscr{D}(J), \quad \mathscr{D}(I) \xleftarrow{f^*} \mathscr{D}(J).$$

- $f_!$ = (homotopy) left Kan extension,
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If $J = *$, then $f^* = \mathrm{const}$, and we get homotopy colimits/limits back.

(Der4) $f_!(X)_j \cong \mathrm{hocolim}_{(f/j)} \mathrm{proj}^*(X)$ and $f_*(X)_j \cong \mathrm{holim}_{(j/f)} \mathrm{proj}^*(X)$ canonically.

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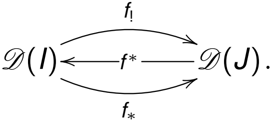

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- Let $\mathscr{V}$ be the derivator of topological spaces, i.e. $\mathcal{C} = \mathcal{T}op$, $W =$ weak equivalences, and $\mathscr{V}(I) = \mathcal{C}^I[W_I^{-1}]$.
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$$\text{“} - \otimes X\text{''}: \mathscr{V}(J) \to \mathscr{D}(I \times J), \quad \text{pt.} \mapsto X.$$

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Outline

1 Homotopy (co)limits

2 Grothendieck derivators

3 Stability

4 Abstract representation theory

Pointed derivators

Definition

A derivator $\mathscr{D}$ is pointed if the base category $\mathscr{D}(*)$ has a zero object (equivalently, each $\mathscr{D}(I)$ has a zero object).

Examples

1. $\mathscr{D}_{Top_*}$, the homotopy derivator of pointed spaces. I.e. $\mathscr{D}_{Top_*}(I)$ is the homotopy category of I-shaped diagrams of pointed topological spaces.
2. $\mathscr{D}_R$ for any ring R. Recall: $\mathscr{D}(I) = \mathrm{D}(\mathrm{Mod}\,R^I)$.

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Suspension and loop functors

Let $\mathscr{D}$ be a pointed derivator. Consider the functors in $\mathcal{C}at$:

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00 \quad \longrightarrow \quad \begin{array}{ccc} 00 & \longrightarrow & 10 \\ \downarrow & & \\ 01 & & \end{array} \quad \longrightarrow \quad \begin{array}{ccc} 00 & \longrightarrow & 10 \\ \downarrow & & \downarrow \\ 01 & \longrightarrow & 11 \end{array} \quad \longleftarrow \quad 11
$$

Then we have functors:

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\mathscr{D}(*) \xrightarrow{00_*} \mathscr{D}(\ulcorner) \xrightarrow{i_\ulcorner} \mathscr{D}(\square) \xrightarrow{11^*} \mathscr{D}(*).
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In terms of diagrams, we have:

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Let $\mathscr{D}$ be a pointed derivator. Consider the functors in $\mathcal{C}at$:

$$\begin{array}{ccccccc} 00 & & \begin{array}{ccc} 00 & \longrightarrow & 10 \\ \downarrow & & \\ 01 & & \end{array} & & \begin{array}{ccc} 00 & \longrightarrow & 10 \\ \downarrow & & \downarrow \\ 01 & \longrightarrow & 11 \end{array} & & \\ & \longrightarrow & & \longrightarrow & & \longleftarrow & \\ & & & & & & 11 \end{array}$$

Then we have functors:

$$\mathscr{D}(*) \xrightarrow{00_*} \mathscr{D}(\ulcorner) \xrightarrow{i_\ulcorner} \mathscr{D}(\square) \xrightarrow{11^*} \mathscr{D}(*).$$

In terms of diagrams, we have:

$$\begin{array}{ccccccc} X & & \begin{array}{ccc} X & \longrightarrow & 0 \\ \downarrow & & \\ 0 & & \end{array} & & \begin{array}{ccc} X & \longrightarrow & 0 \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & \Sigma X \end{array} & & \\ & \longmapsto & & \longmapsto & & \longmapsto & \\ & & & & & & \Sigma X \end{array}$$

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Definition

A pointed derivator $\mathscr{D}$ is stable if (Σ, Ω) is a pair of equivalences.

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Equivalently: pullbacks and pushouts coincide in $\mathscr{D}(\square)$.

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- The homotopy derivator $\mathscr{D}_{\mathcal{S}p}$ of topological spectra.
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Additivity and triangulated structure

Theorem (Franke, Maltsiniotis, Groth)

A stable derivator admits a canonical additive structure, i.e. we actually have a 2-functor

$$\mathscr{D}\colon \mathcal{C}at^{\mathrm{op}} \to \mathrm{ADD}.$$

Under an additional mild hypothesis, we even have a canonical triangulated structure:

$$\mathscr{D}\colon \mathcal{C}at^{\mathrm{op}} \to \mathrm{TRIA}.$$

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Outline

1. Homotopy (co)limits

2. Grothendieck derivators

3. Stability

4. Abstract representation theory

Derivators vs. representation theory

- Classically representation theory is concerned with studying modkA, where k is a field and $A \in \mathcal{C}at$.
- More modern version: study D(kA), the derived category. But D(kA) is none other than $\mathscr{D}_k(A)$.

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Abstract representation theory

- If $\mathscr{D}$ is any stable derivator, we can view $\mathscr{D}(I)$ as the "derived" category of representations of I in $\mathscr{D}$.
- For instance, given the derivator $\mathscr{D}_{\mathcal{S}p}$ of spectra, $\mathscr{D}_{\mathcal{S}p}(I)$ is the homotopy category of I-shaped diagrams of spectra (universal example).
- The point: Various familiar patterns from representation theory apply to any stable derivator.
- Applications:
  - Equivalences via Bernstein-Gelfand-Ponomarev reflection functors.
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Reflection functors for Dynkin type A

- Let $\mathscr{D}$ be any stable derivator and $X \in \mathscr{D}([n])$ (of shape $X_0 \to X_1 \to \cdots \to X_n$).
- By a series of Kan extension construct a coherent diagram of the following shape with all squares bicartesian (n = 2):

- Restrict to a suitable part of the diagram to obtain equivalences or autoequivalences.

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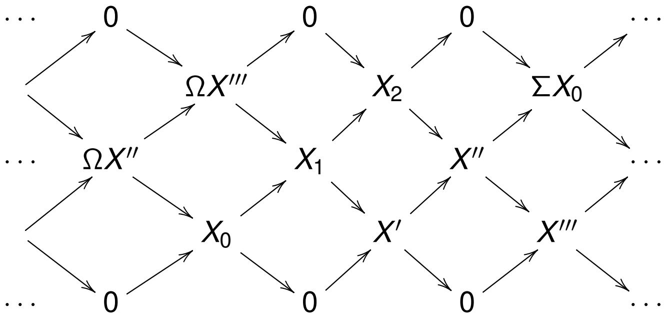

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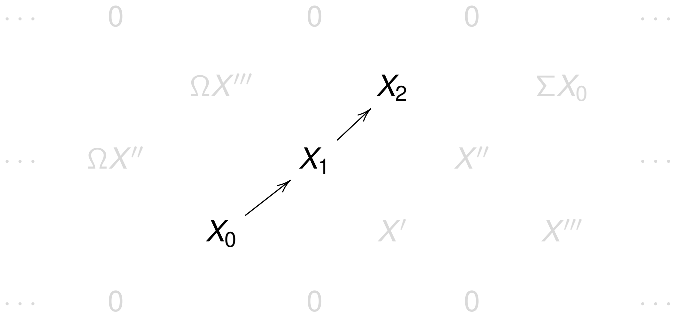

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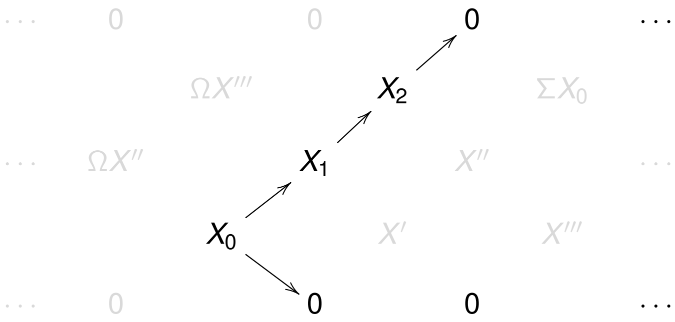

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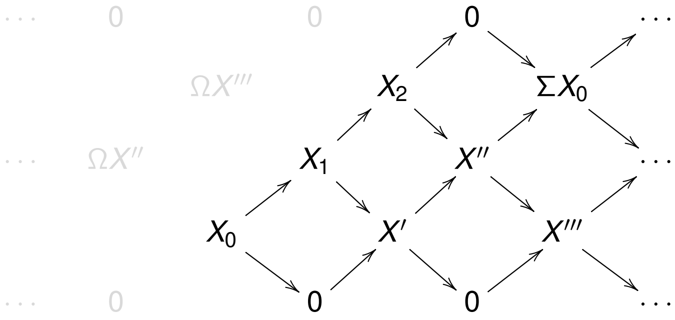

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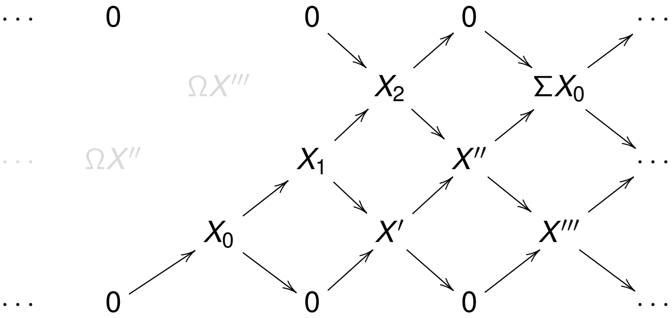

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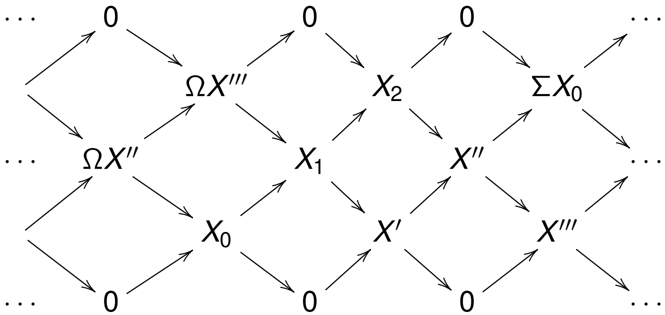

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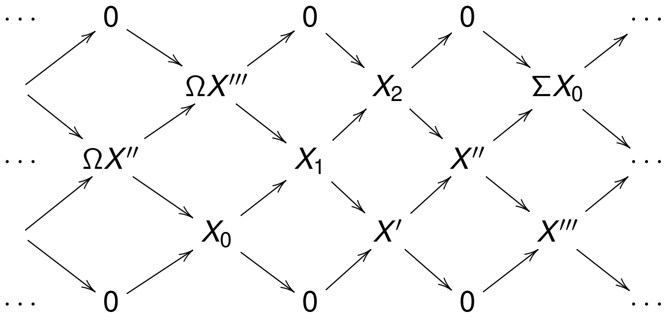

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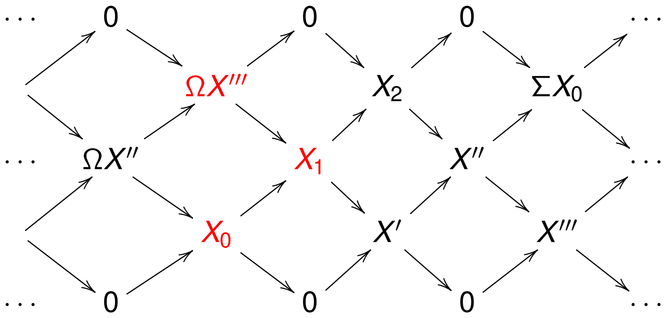

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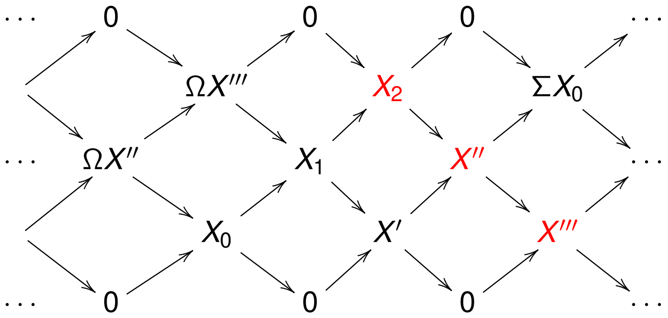

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Universal APR tilting “modules”

- We can always obtain an object $T \in \mathscr{D}_{\mathcal{S}p}([n] \times [n]^{\mathrm{op}})$ such that these (auto)equivalences are of the form

$$T \otimes_{[[n]]} - : \mathscr{D}([n]) \to \mathscr{D}([n]).$$

- Here, $T \otimes_{[[n]]} X = \int^{[n]} T \otimes X$ (the coend).
- If k is a field then $T \otimes_{[[n]]} k \in \mathscr{D}([n]) = \mathrm{D}(k[n])$ is a classical tilting module.

Example

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$$\mathbb{I}\colon \quad \begin{array}{ccccc} \mathbb{S} & \to & \mathbb{S} & \to & \mathbb{S} \\ \uparrow & & \uparrow & & \uparrow \\ 0 & \to & \mathbb{S} & \to & \mathbb{S} \\ \uparrow & & \uparrow & & \uparrow \\ 0 & \to & 0 & \to & \mathbb{S} \end{array} \qquad T\colon \quad \begin{array}{ccccc} \mathbb{S} & \to & \mathbb{S} & \leftarrow & 0 \\ \uparrow & & \uparrow & & \uparrow \\ 0 & \to & \mathbb{S} & \leftarrow & 0 \\ \uparrow & & \uparrow & & \uparrow \\ 0 & \to & 0 & \leftarrow & \Omega\mathbb{S} \end{array}$$

Universal APR tilting "modules"

- We can always obtain an object $T \in \mathscr{D}_{\mathcal{S}p}([n] \times [n]^{\mathrm{op}})$ such that these (auto)equivalences are of the form

$$T \otimes_{[[n]]} - \colon \mathscr{D}([n]) \to \mathscr{D}([n]).$$

- Here, $T \otimes_{[[n]]} X = \int^{[n]} T \otimes X$ (the coend).
- If k is a field then $T \otimes_{[[n]]} k \in \mathscr{D}([n]) = \mathrm{D}(k[n])$ is a classical tilting module.

Example

$$\mathbb{I}\colon \quad \begin{array}{ccccc} \mathbb{S} & \to & \mathbb{S} & \to & \mathbb{S} \\ \uparrow & & \uparrow & & \uparrow \\ 0 & \to & \mathbb{S} & \to & \mathbb{S} \\ \uparrow & & \uparrow & & \uparrow \\ 0 & \to & 0 & \to & \mathbb{S} \end{array} \qquad\qquad T\colon \quad \begin{array}{ccccc} \mathbb{S} & \to & \mathbb{S} & \leftarrow & 0 \\ \uparrow & & \uparrow & & \uparrow \\ 0 & \to & \mathbb{S} & \leftarrow & 0 \\ \uparrow & & \uparrow & & \uparrow \\ 0 & \to & 0 & \leftarrow & \Omega\mathbb{S} \end{array}$$