NTNU
Norwegian University of
Science and Technology

Categorification—the $\mathfrak{sl}_2$ case

Functor valued invariants of knots, links and tangles

Jan Šťovíček
Department of Mathematical Sciences

October 22^{nd}, 2009

Outline

1. Knots and links
2. The tangle category and quantum groups
3. Categorification
4. Lie algebras and the category $\mathcal{O}$

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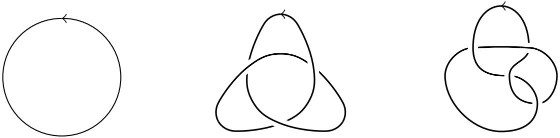

Unknot Trefoil Knot 4_1

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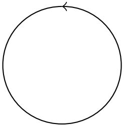

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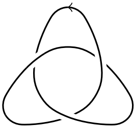

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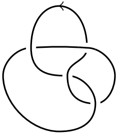

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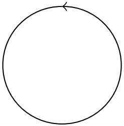

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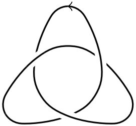

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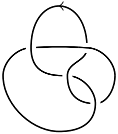

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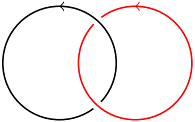
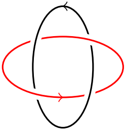


Hopf link



Solomon's knot

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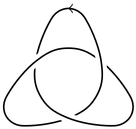

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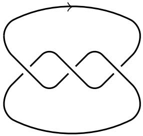

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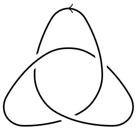

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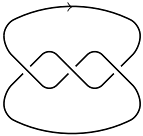

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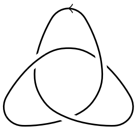

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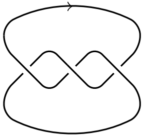

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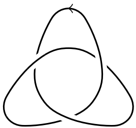

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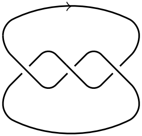

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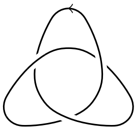

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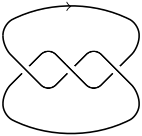

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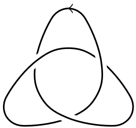

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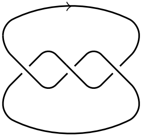

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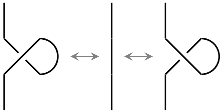
- algebraic invariants—Jones polynomial, functors.

Reidemeister moves



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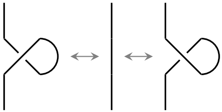

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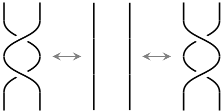

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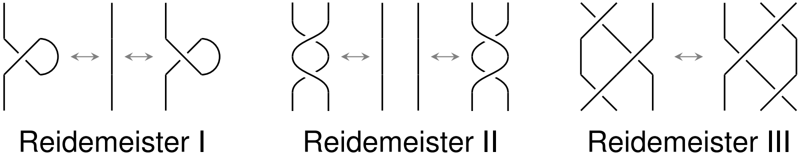

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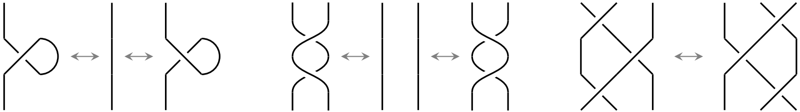

Reidemeister I   Reidemeister II   Reidemeister III

Theorem (Reidemeister 1926, Alexander and Briggs 1927)

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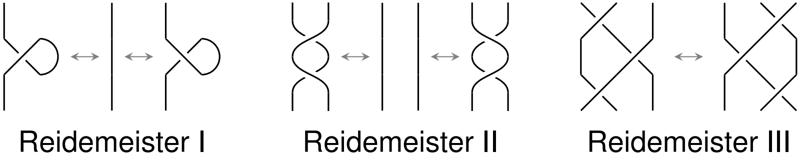

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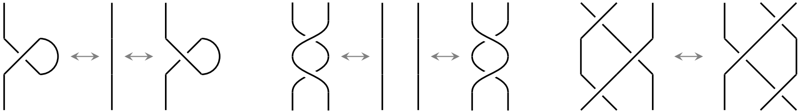

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Theorem (Reidemeister 1926, Alexander and Briggs 1927)

Two knots are isotopic if and only if their diagrams can be transferred to each other using planar isotopy and the Reidemeister moves I–III above.

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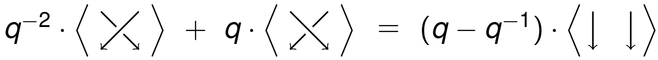

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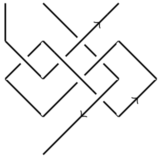

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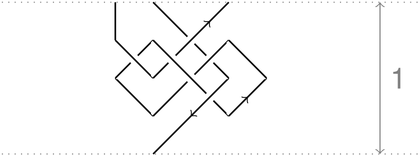

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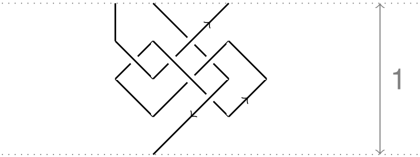

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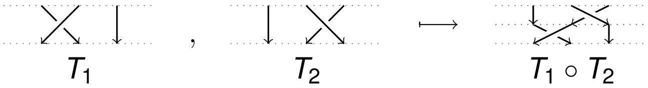

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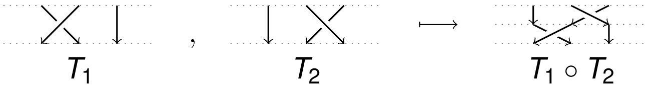

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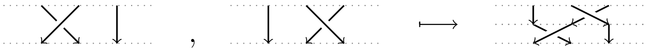

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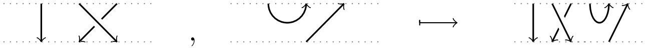

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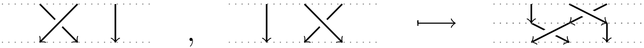

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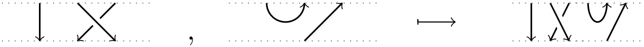

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— Important: These operations are compatible with isotopies!

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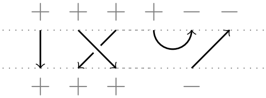

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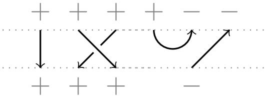

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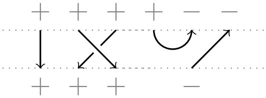

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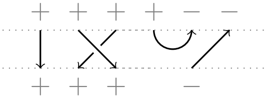

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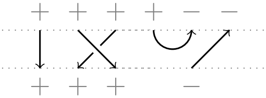

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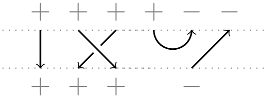

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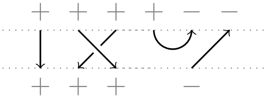

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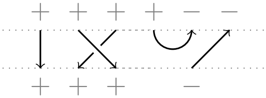

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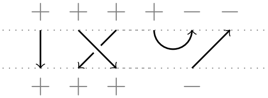

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— Indeed, we can reconstruct the Jones polynomial by making a particular choice of F for $k = 2$!

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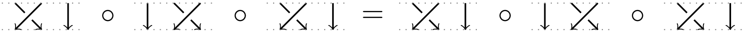

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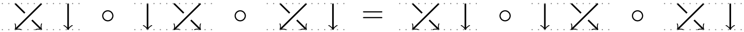

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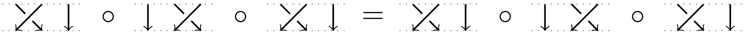

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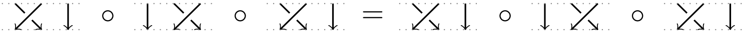

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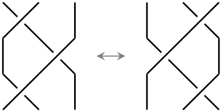
— Based on work of Turaev and Yetter, around 1988.

YB-equation and quantum groups

— Put $\Phi = F($$)$, recall Reidemeister III: 

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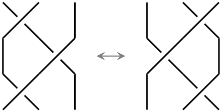

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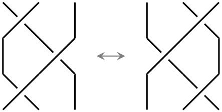

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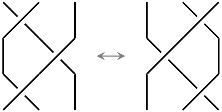

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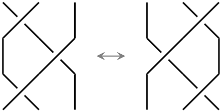

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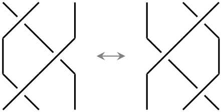
— Put $\Phi = F(\times)$, recall Reidemeister III: 

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Outline

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Outline

1. Knots and links
2. The tangle category and quantum groups
3. Categorification
4. Lie algebras and the category $\mathcal{O}$

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— A weight space W_α corresponding to $\alpha : \mathfrak{h} \to \mathbb{C}$ is defined as $W_\alpha = \{w \in W \mid H \cdot w = \alpha(H)w \text{ for each } H \in \mathfrak{h}\}$.

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Legend:

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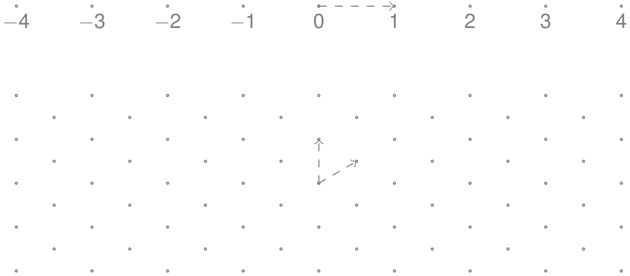

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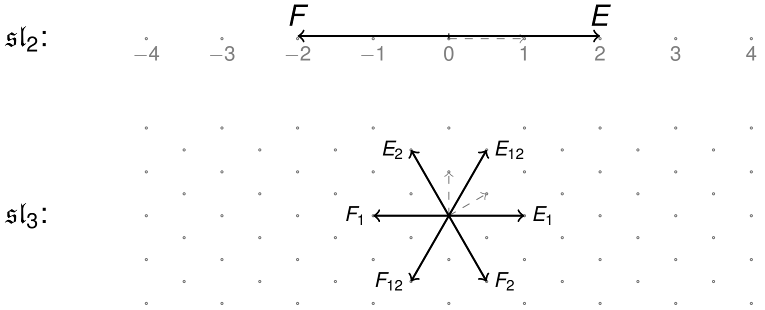

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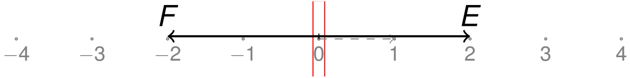

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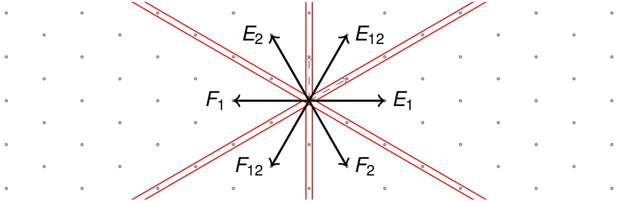

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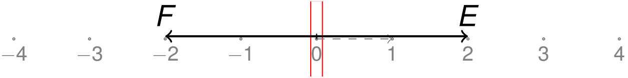

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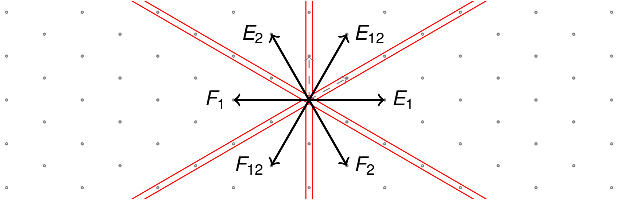

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Components delimited by walls are called chambers.

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— To be precise, there are a few more technical aspects

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